Volume 4, No. 10, Sep-Oct 2013

ISSN No. 0976-5697

International Journal of Advanced Research in Computer Science

RESEARCH PAPER

Available Online at www.ijarcs.info

A Theory of Fs-sets,Fs-Complements and Fs-De Morgan Laws

Vaddiparthi Yogeswara*
Dept. Mathematics, GIT
GITAM University
Visakhapatnam-530045,A.P State, India
vaddiparthyy@yahoo.com

Gudivada Srinivasa Rao
Research Scholar : Dept. of Mathematics GIT
GITAM University
Visakhapatnam-530045,A.P State, India
vasumsc11@.gmail.com

Biswajit Rath
Dept. of Mathematics
GIACR Engg. college
Rayagada-765002, Odisha State,India
urwithbr@gmail.com

Abstract: In this paper we introduced Fs-set, Fs-subset etc and we define Fs-complement and prove De Morgan laws of Fs-subsets.

Keywords: Fs-set, Fs-subset, Fs-empty set, Fs-union, Fs-intersection, Fs-complement and Fs-De Morgan laws.

I. INTRODUCTION

Murthy[1] introduced F-set in order to prove Axiom of choice for fuzzy sets which is not true for L-fuzzy sets introduced by Goguen[2]. In the paper[3], Tridiv discussed fuzzy complement of an extended fuzzy subset and proved De Morgan laws etc. The extended Fuzzy sets Tridiv considered contains the membership value $\mu_1(x)-\mu_2(x)$. $-\mu_2(x)$, a term is in this expression will not be in the interval [0,1].Also they discussed similar results in [4].To answer this incomprehensiveness, we introduced the concept of Fs-set and developed the theory of Fs-sets in this paper . The object of this theory is to introduce Fs-complement of a Fs -subset similar to fuzzy complement of a fuzzy set, so that the De Morgan laws which are called the Fs- De Morgan laws in the new theory are to be proved.

The membership values of Fs-set and Fs-subset lie in a complete Boolean algebra[5] and we define Fs-union, Fs-intersection, Fs-complement and proved collection of all Fs-subsets is a complete lattices under Fs-union and Fs-intersection. We denote Fs-union and crisp set union by same symbol ∪ and similary Fs-intersection and crisp set intersection by the same symbol ∩.Distribution laws hold partially . We stated Fs-De Morgan laws and proved one of the Fs-De Morgan laws is true [2.7(ii)] and other Fs-De Morgan law was conditionally true[2.7(i)]. We denote the largest element of a complete Boolean algebra L_A[1.1] byM_A,the complement of b in L_A by b^c. For any crisp subset B, the usual set complement of B , is denoted by B^c and $B^c\cup A$ is denoted by C_AB .Complete Boolean algebras in this paper are generally represented by suitable diagrams. .For all lattice theoretic properties and Boolean algebraic properties we refer Szasz [6], Garret Birkhoff[7],Steven Givant • Paul Halmos[5] and Thomas Jech[8]

II. THEORY OF FS-SETS

A. *Fs-set:*

Let U be a universal set, $A_1\subseteq U$ and let $A\subseteq U$ be non-empty. A four tuple $\mathcal{A}=\left(A_1,A,\bar{A}\left(\mu_{1A_1},\mu_{2A}\right),L_A\right)$ is said be an Fs-set if, and only if

(1) $A\subseteq A_1$
(2) L_A is a complete Boolean Algebra
(3) $\mu_{1A_1}:A_1\longrightarrow L_A$, $\mu_{2A}:A\longrightarrow L_A$,are such that $\mu_{1A_1}|A\geq\mu_{2A}$
(4) $\bar{A}$:A⟶ L_A is defined by $\bar{A}x=\mu_{1A_1}x\wedge(\mu_{2A}x)^c$,for each $x\in A$

B. *Fs-subset*

Let $\mathcal{A}$=$\left(A_1,A,\bar{A}\left(\mu_{1A_1},\mu_{2A}\right),L_A\right)$ and $\mathcal{B}$=$\left(B_1,B,\bar{B}\left(\mu_{1B_1},\mu_{2B}\right),L_B\right)$ be a pair of Fs-sets. $\mathcal{B}$ is said to be an Fs-subset of $\mathcal{A}$, denoted by $\mathcal{B}\subseteq\mathcal{A}$, if, and only if

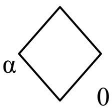
a. $B_1\subseteq A_1$, $A\subseteq B$
b. L_B is a complete subalgebra of L_A or $L_B\leq L_A$
c. $\mu_{1B_1}\leq\mu_{1A_1}|B_1$, and $\mu_{2B}|A\geq\mu_{2A}$

C. *Proposition:*

Let $\mathcal{B}$ and $\mathcal{A}$ be a pair of Fs-sets such that $\mathcal{B}\subseteq\mathcal{A}$. Then $\bar{B}x\leq\bar{A}x$ is true for each $x\in A$

The proof follows from the definitions of Fs-subset, $\bar{B}x$ and $\bar{A}x$.

a. *Example: 1*

Let $A_1=\{a_1,a_2\}$, $A=\{a_1\}$, L_A = α  β
0 Fig-I

$\mu_{1A_1}(a_1)=1$ and $\mu_{1A_1}(a_2)=0=\mu_{2A}(a_1)$

$\therefore$ $\bar{A}a_1=\mu_{1A_1}(a_1)\wedge(\mu_{2A}(a_1))^c=1\wedge0^c=1\wedge1=1$

Then $\mathcal{A}=\left(A_1,A,\bar{A}\left(\mu_{1A_1},\mu_{2A}\right),L_A\right)$ is an Fs-set

Again suppose $B_1=\{a_1\}$, $B=\{a_1\}$, $L_B=L_A$, $\mu_{1B_1}(a_1)=1$, $\mu_{2B}(a_1)=0$

$\therefore$ $\bar{B}(a_1)=\mu_{1B_1}(a_1)\wedge(\mu_{2B}(a_1))^c=1\wedge0^c=1\wedge1=1$

Then $\mathcal{B}$= $\left(B_1,B,\bar{B}\left(\mu_{1B_1},\mu_{2B}\right),L_B\right)$ is an Fs-subset of$\mathcal{A}$

D. Definition:

For some L_X, such that $L_X \leq L_A$ a four tuple $\mathcal{X} = (X_1, X, \bar{X}(\mu_{1X_1}, \mu_{2X}), L_X)$ is not an Fs-set if, and only if

(a) $X \nsubseteq X_1$ or

(b) $\mu_{1X_1}x \ngeq \mu_{2X}x$, for some $x \in X \cap X_1$

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of $\mathcal{B}$ for any $\mathcal{B} \subseteq \mathcal{A}$.

Definition: An Fs-subset $\mathcal{Y}=(Y_1, Y, \bar{Y}(\mu_{1Y_1}, \mu_{2Y}), L_Y)$ of $\mathcal{A}$, is said to be an Fs-empty set of second kind if, and only if

(a') $Y_1 = Y = A$

(b') $L_Y \leq L_A$

(c') $\bar{Y} = 0$

a. Remark:

We denote Fs-empty set of first kind or Fs-empty set of second kind by $\Phi_{\mathcal{A}}$ and we prove later (1.15), $\Phi_{\mathcal{A}}$ is the least Fs-subset among all Fs-subsets of $\mathcal{A}$.

E. Definition of equality of two Fs-sets:

Let $\mathcal{B}_1 = (B_{11}, B_1, \bar{B}_1(\mu_{1B_{11}}, \mu_{2B_1}), L_{B_1})$ and $\mathcal{B}_2 = (B_{12}, B_2, \bar{B}_2(\mu_{1B_{12}}, \mu_{2B_2}), L_{B_2})$ be a pair of Fs-sets. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal, denoted by $\mathcal{B}_1 = \mathcal{B}_2$ if, only if

(1) $B_{11} = B_{12}$, $B_1 = B_2$

(2) $L_{B_1} = L_{B_2}$

(3) (a) $(\mu_{1B_{11}} = \mu_{1B_{12}}$ and $\mu_{2B_1} = \mu_{2B_2})$,or (b) $\bar{B}_1 = \bar{B}_2$

a. Remark:

We can easily observed that 3 (a) and 3 (b) not equivalent statements.

b. Example:

Let $\mathcal{A}=(A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, where $A_1 = \{a, b, c\}, A = \{a\}$,where $L_A = L_B$ is the fig-II. 1

$\mu_{1A_1}: A_1 \longrightarrow L_A$ is given by $\mu_{1A_1} = 1$

$\mu_{2A}: A \longrightarrow L_A$ is given by $\mu_{2A} = 0$

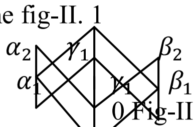


Fig-II

$\bar{A}: A \longrightarrow L_A$ is given by, $\bar{A}x = \mu_{1A_1}x \wedge (\mu_{2A}x)^c = 1 \wedge 0^c = 1$

$\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$

$B_1 = \{a, b\}, B = \{a\}$, $L_B = L_A$

$\mu_{1B_1}: B_1 \longrightarrow L_B$ is given by $\mu_{1B_1} = \alpha_2$

$\mu_{2B}: B \longrightarrow L_B$ is given by $\mu_{2B} = \alpha_1$

$\bar{B}: B \longrightarrow L_B$ is given by $\bar{B}x = \mu_{1B_1}x \wedge (\mu_{2B}x)^c = \alpha_2 \wedge (\alpha_1)^c = \alpha_2 \wedge \beta_2 = \gamma_1$

$\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ where $C_1 = \{a, b\}, C = \{a\}$, $L_c = L_A$

$\mu_{1C_1}: C_1 \longrightarrow L_C$ is given by $\mu_{1C_1} = \beta_2$

$\mu_{2C}: C \longrightarrow L_C$ is given by $\mu_{2C} = \beta_1$

$\bar{C}: C \longrightarrow L_C$ is given by $\bar{C}x = \mu_{1C_1}x \wedge (\mu_{2C}x)^c = \beta_2 \wedge (\beta_1)^c = \beta_2 \wedge \alpha_2 = \gamma_1$

We can observed that

$\mu_{1B_1} \neq \mu_{1C_1}$ and $\mu_{2B} \neq \mu_{2C}$ but $\bar{B}$=$\bar{C}$

F. Proposition:

$\mathcal{B}_1 = (B_{11}, B_1, \bar{B}_1(\mu_{1B_{11}}, \mu_{B_1}), L_{B_1})$ and $\mathcal{B}_2 = (B_{12}, B_2, \bar{B}_2(\mu_{1B_{12}}, \mu_{B_2}), L_{B_2})$ are equal if, only if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$

***Proof:* (⇒):** Part of the proposition.

Let $\mathcal{B}_1 = \mathcal{B}_2$. Then we have the following

(i) $B_{11} = B_{12}$, $B_1 = B_2$

(ii) $L_{B_1} = L_{B_2}$

(iii) (a) $(\mu_{1B_{11}} = \mu_{1B_{12}}, \ \mu_{2B_1} = \mu_{2B_2})$ or (b) $\bar{B}_1 = \bar{B}_2$

$\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$ follow from (i),(ii)and(iii)

(⇐): Part of the proposition.

Suppose $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$. Then we have the following

(1) $B_{11} \subseteq B_{12}$ and $B_1 \supseteq B_2$

(2) $L_{B_1} \leq L_{B_2}$

(3) $\mu_{1B_{11}}x \leq \mu_{1B_{12}}x$, for each $x \in B_{11}$, $\mu_{2B_1}x \geq \mu_{2B_2}x$ for each $x \in B_2$

And

(1') $B_{12} \subseteq B_{11}$ and $B_2 \supseteq B_1$

(2') $L_{B_2} \leq L_{B_1}$

(3') $\mu_{1B_{12}}x \leq \mu_{1B_{11}}x$, for each $x \in B_{12}$, $\mu_{2B_2} \geq \mu_{2B_1}$, for each $x \in B_1$

(d') $B_{11} = B_{12}$ and $B_1 = B_2$ follow from (1) and (1')

(e') $L_{B_1} = L_{B_2}$, follows from (2) and (2')

(f') $(\mu_{1B_{11}} = \mu_{1B_{12}}$ and $\mu_{2B_1} = \mu_{2B_2})$ or $\bar{B}_2 = \bar{B}_1$, follow from (3) and (3')

Hence $\mathcal{B}_1 = \mathcal{B}_2$ follow from (d'),(e') and (f')

G. Definition of Fs-union for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C}= (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$, be a pair of Fs-subsets of $\mathcal{A}$. Then,

the Fs-union of $\mathcal{B}$ and $\mathcal{C}$,denoted by $\mathcal{B} \cup \mathcal{C}$ is defined as $\mathcal{B} \cup \mathcal{C}$=$\mathcal{D}$=$(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$,where

a. $D_1 = B_1 \cup C_1$, $D = B \cap C$

b. $L_D = L_B \vee L_C$=complete subalgebra generated by $L_B \cup L_C$

c. $\mu_{1D_1}: D_1 \longrightarrow L_D$ is defined by $\mu_{1D_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$

$\mu_{2D} : D \longrightarrow L_D$ is defined by $\mu_{2D}x = \mu_{2B}x \wedge \mu_{2C}x$ and $\bar{D}$: $D \longrightarrow L_D$ is defined by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c$

H. Proposition:

$\mathcal{B} \cup \mathcal{C}$ is an Fs-subset of $\mathcal{A}$.

The prove directly follows from the definition of $\mathcal{B} \cup \mathcal{C}$.

I. Definition of Fs-intersection for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C}= (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ be a pair of Fs-subsets of $\mathcal{A}$ satisfying the following conditions:

(i) $B_1 \cap C_1 \supseteq B \cup C$

(ii) $\mu_{1B_1}x \wedge \mu_{1C_1}x \geq (\mu_{2B} \vee \mu_{2C})x$, for each $x \in A$

Then, the Fs-intersection of $\mathcal{B}$ and $\mathcal{C}$, denoted by $\mathcal{B} \cap \mathcal{C}$ is defined as

$$\mathcal{B} \cap \mathcal{C} = \mathcal{E} = (E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E),\text{where}$$

(a) $E_1 = B_1 \cap C_1$, $E = B \cup C$

(b) $L_E = L_B \wedge L_C$=$L_B \cap L_C$

(c) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is defined by $\mu_{1E_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$

$\mu_{2E} : E \longrightarrow L_E$ is defined by $\mu_{2E}x = (\mu_{2B} \vee \mu_{2C})x$

$\bar{E}$: $E \longrightarrow L_E$ is defined by $\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$.

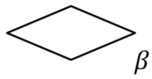
a. Remark:

If (i) or (ii) fails we define $\mathcal{B} \cap \mathcal{C}$ as $\mathcal{B} \cap \mathcal{C}$=$\Phi_{\mathcal{A}}$, which is the Fs-empty set of first kind.

b. Example:

Let $\mathcal{A} = (A_1, A, \bar{A}\,(\mu_{1A_1}, \mu_{2A}), L_A)$, where

$A_1 = \{a, b, c\}, A = \{a\}$



μ_{1A_1}:$A_1 \longrightarrow L_A$ is given by $\mu_{1A_1} = 1$, $L_A = \alpha$ β

$\mu_{2A}: A \longrightarrow L_A$ is given by $\mu_{2A} = 0$ 0 fig-III

$\bar{A}(a) = \mu_{1A_1}(a) \wedge (\mu_{2A}(a))^c$ = 1∧ 0^c =1

Let $\mathcal{B}$=$(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, where

$B_1 = \{a, b, c\}, B = \{a\}$, L_B= L_A

$\mu_{1B_1}: B_1 \longrightarrow L_B$ is given by $\mu_{1B_1}(a) = \alpha, \mu_{1B_1}(b) = 1, \mu_{1B_1}(c) = \beta$

$\mu_{2B}: B \longrightarrow L_B$ is given by $\mu_{2B}(a) = \beta$

$\bar{B}: B \longrightarrow L_B$, is given by $\bar{B}(a) = \mu_{1B_1}(a) \wedge (\mu_{2B}(a))^c$ =1∧ β^c =α

Let $\mathcal{C}$= $(\mathrm{C}_1, \mathrm{C}, \bar{\mathrm{C}}(\mu_{1C_1}, \mu_{2C}), L_C)$, where$C_1 = \{a, c\}, C = \{a\}$, L_C= L_A

$C_{1C_1}: C_1 \longrightarrow L_C$ is given by $\mu_{1C_1} = \beta$

$\mu_{2C}: C \longrightarrow L_C$ is given by $\mu_{2C} = 0$

$\bar{C}: C \longrightarrow L_C$ is given by $\bar{C}(a) = \mu_{1C_1}(a) \wedge (\mu_{2C}(a))^c$ =β∧ 0^c =β

<u>Fs-union of $\mathcal{B}$ and $\mathcal{C}$</u>

Let $\mathcal{B}$∪$\mathcal{C}$ =$\mathcal{D}$=$(\mathrm{D}_1, \mathrm{D}, \bar{\mathrm{D}}(\mu_{1D_1}, \mu_{2D}), L_D)$,where

$D_1 = B_1 \cup C_1 = \{a, b, c\} \cup \{a, c\} = \{a, b, c\}$,$B \cap C$=$\{a\}$,

$L_D = L_B \vee L_C$= L_A

$\mu_{1D_1}: D_1 \longrightarrow L_D$ is given by $\mu_{1D_1}(a) = \mu_{1B_1}(a) \vee \mu_{1C_1}(a) = 1 \vee \beta = 1$

$\mu_{1D_1}(c) = \mu_{1B_1}(c) \vee \mu_{1C_1}(c) = \beta \vee \beta = \beta$

$\mu_{2D} : D \longrightarrow \mathrm{L}_D$ is given by$\mu_{2D}(a) = \mu_{2B}(a) \wedge \mu_{2C}(a) = \beta \wedge 0 = 0$ and

$\bar{D}: D \longrightarrow L_D$ is given by $\bar{D}(a) = \mu_{1D_1}(a) \wedge (\mu_{2D}(a))^c = 1 \wedge 0^c = 1$

∴ $\mathcal{B}$∪$\mathcal{C}$ =$\mathcal{D}$=$(\{a, b, c\}, \{a\}, \bar{D}(1,0), L_A)$

<u>Fs-intersection of $\mathcal{B}$ and $\mathcal{C}$</u>

Let $\mathcal{B}$∩$\mathcal{C}$ =$\mathcal{E}$=$(E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$,where

$E_1 = B_1 \cap C_1 = \{a, b, c\} \cap \{a, c\} = \{a, c\}, E = B \cup C = \{a\}$

$L_E = L_B \wedge L_C = L_A$

$\mu_{1E_1}: E_1 \longrightarrow L_E$ is defined by $\mu_{1E_1}(a) = \mu_{1B_1}(a) \wedge \mu_{1C_1}(a) = 1 \wedge \beta = \beta$

$\mu_{1E_1}(c) = \mu_{1B_1}(c) \wedge \mu_{1C_1}(c) = 1 \wedge \beta = \beta$

$\mu_{2E} : E \longrightarrow \mathrm{L}_E$ is defined by $\mu_{2E}(a) = \mu_{2B}(a) \vee \mu_{2C}(a) = \beta \wedge 0 = 0$

$\bar{E}: E \longrightarrow L_E$ is defined by,$\bar{E}(a) = \mu_{1E_1}(a) \wedge (\mu_{2E}(a))^c = \beta \wedge \beta^c = 0$

Here we observed that

(i) $B_1 \cap C_1 \supseteq B \cup C$

(ii) $\mu_{1B_1}x \wedge \mu_{1C_1}x \geq (\mu_{2B} \vee \mu_{2C})x$, for each $x \in \mathrm{B} \cup \mathrm{C}$

∴ $\mathcal{B}$∩$\mathcal{C}$ =$\mathcal{E}$=$(\{a, c\}, \{a\}, \bar{E}(\beta, \beta), L_A)$.

J. Proposition:

For any pair of Fs-subsets $\mathcal{B}$=$(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$and $\mathcal{C}$=$(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ of$\mathcal{A}$, the following results are true

a. $\mathcal{B}$⊆$\mathcal{B}$∪$\mathcal{C}$ and $\mathcal{C}$⊆$\mathcal{B}$∪$\mathcal{C}$
b. $\mathcal{B}$∩$\mathcal{C}$⊆$\mathcal{B}$ and $\mathcal{B}$∩$\mathcal{C}$⊆$\mathcal{C}$ provided $\mathcal{B}$∩$\mathcal{C}$ exists
c. $\mathcal{B}$⊆$\mathcal{C}$ implies $\mathcal{B}$∪$\mathcal{C}$=$\mathcal{C}$
d. $\mathcal{B}$∩$\mathcal{C}$=$\mathcal{B}$ when $\mathcal{B}$≠ $\Phi_{\mathcal{A}}$ and $\mathcal{B}$⊆$\mathcal{C}$ and $\Phi_{\mathcal{A}}$ ∩$\mathcal{C}$=$\Phi_{\mathcal{A}}$
e. $\mathcal{B}$∪$\mathcal{C}$= $\mathcal{C} \cup \mathcal{B}$ (commutative law of Fs-union)
f. $\mathcal{B}$∩$\mathcal{C}$= $\mathcal{C} \cap \mathcal{B}$ provided $\mathcal{B}$∩$\mathcal{C}$ exists. (commutative law of Fs-intersection)
g. $\mathcal{B}$∪$\mathcal{B}$=$\mathcal{B}$
h. $\mathcal{B}$∩$\mathcal{B}$=$\mathcal{B}$ ((7)and (8) are Idempotent laws of Fs-union and Fs-intersection respectively)

***Proof*(1):**Let $\mathcal{B} \cup \mathcal{C} = \mathcal{D} = (\mathrm{D}_1, \mathrm{D}, \bar{\mathrm{D}}(\mu_{1D_1}, \mu_{2D}), L_D)$

(1a) $D_1 = B_1 \cup C_1$, $D = B \cap C$

(1b) $L_D = L_B \vee L_C$

(1c) $\mu_{1D_1}: D_1 \longrightarrow L_D$ is given by $\mu_{1D_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$

$\mu_{2D} : D \longrightarrow \mathrm{L}_D$ is given by$\mu_{2D}x = \mu_{2B}x \wedge \mu_{2C}x$

$\bar{D}: D \longrightarrow L_D$ is given by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c$

We can eas ily s how t hat t he f ollowing ar e t he consequences of (1a),(1b)and(1c)

(1d) $B_1 \subseteq D_1$, $D \subseteq B$

(1e) $\mathrm{L_B} \leq \mathrm{L_D}$

(1f) $\mu_{1B_1} \leq \mu_{1D_1}|B_1$, and $\mu_{2B}|D \geq \mu_{2D}$

These in term imply $\mathcal{B}$⊆$\mathcal{B}$∪$\mathcal{C}$

Similarly we can prove that $\mathcal{C}$⊆$\mathcal{B}$∪$\mathcal{C}$

Proof(2):Let $\mathcal{B}$∩$\mathcal{C}$=$\mathcal{E}$=$(\mathrm{E}_1, \mathrm{E}, \bar{\mathrm{E}}(\mu_{1E_1}, \mu_{2E}), L_E)$,where

(2a) $E_1 = B_1 \cap C_1$, $E = B \cup C$

(2b) $L_E = L_B \wedge L_C$

(2c) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is given by $\mu_{1E_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$

$\mu_{2E} : E \longrightarrow \mathrm{L}_E$ is given by $\mu_{2E}x = (\mu_{2B} \vee \mu_{2C})x$

$\bar{E}: E \longrightarrow L_E$ is given by $\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$

We can eas ily s how t hat t he f ollowing ar e t he consequences of (2a),(2b) and (2c) and existence of $\mathcal{B}$∩$\mathcal{C}$

(2d) $E_1 \subseteq B_1$, $E \subseteq B$

(2e) $\mathrm{L_E} \leq \mathrm{L_B}$

(2f) $\mu_{1E_1} \leq \mu_{1B_1}|E_1$, and $\mu_{2E}|B \geq \mu_{2B}$

Hence $\mathcal{B}$∩$\mathcal{C}$⊆$\mathcal{B}$

Similarly we can prove that $\mathcal{B}$∩$\mathcal{C}$⊆$\mathcal{C}$

Proof (3): The following are true since $\mathcal{B}$⊆ $\mathcal{C}$.

(3a) $B_1 \subseteq C_1$, $C \subseteq B$

(3b) $\mathrm{L_B} \leq \mathrm{L_C}$

(3c) $\mu_{1B_1} \leq \mu_{1C_1}|B_1$, and $\mu_{2B}|C \geq \mu_{2C}$

$\mathcal{B}$∪$\mathcal{C}$ =$\mathcal{D}$=$(\mathrm{D}_1, \mathrm{D}, \bar{\mathrm{D}}(\mu_{1D_1}, \mu_{2D}), L_D)$,where

$D_1, D, \bar{D}$ and L_D are as in(1a),(1b)and(1c).

The following are the consequences of(1a),(1b),(3a)and(3b)

(3a') $D_1 = C_1$ and $D = C$

(3b') $L_D = L_c$

We prove

(3c') $\bar{D} = \bar{C}$ from (1c) and (3c)

$\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c$

$=(\mu_{1B_1} \vee \mu_{1C_1})x \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c$

$$=\begin{cases} (\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c, x \in B_1 = B_1 \cap C_1 \\ (\mu_{1C_1}x) \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c, x \notin B_1, x \in C_1 \end{cases}$$

$=(\mu_{1C_1}x) \wedge (\mu_{2C}x)^c$

$=\bar{C}x$, for each $x \in D = C$

Proof (4): From definition of $\mathcal{B}$∩$\mathcal{C}$ and hypothesis of $\mathcal{B}$⊆$\mathcal{C}$, we have

(4a) $E_1 = B_1 \cap C_1 = B_1$,$E = B \cup C = B$ and $E_1 \supseteq E$

(4b) $L_E = L_B \wedge L_C = L_B$

We can observe that

(4c) $\mu_{1E_1}x$ $=\mu_{1B_1}x \wedge \mu_{1C_1}x = \mu_{1B_1}x$, for each $x \in E_1 = B_1$ and

$\mu_{2E}x = (\mu_{2B} \vee \mu_{2C})x$

$$=\begin{cases} \mu_{2B}x & , x \in B, x \notin C \\ \mu_{2B}x \vee \mu_{2C}x = \mu_{2B}x & , x \in B \cap C = C \end{cases}$$

In both cases, we can have $\mu_{1E_1}x \geq \mu_{2E}x$

Hence t he ex istence $\mathcal{B}\cap\mathcal{C}$ is a co nsequence from (4a),(4b) and (4c).

We prove that $\mathcal{B}\cap\mathcal{C}=\mathcal{B}$, that is, $\mathcal{E}=\mathcal{B}$, where $\mathcal{E}$ is as in (2a) , (2b) and (2c).

From(4a) and (4b), we can have

$E_1 = B_1$,$E = B$ and $L_E = L_B$

Sufficient to show that $\bar{E}x = \bar{B}x$ for each $x \in B$

From (2c)

$\bar{E}x = \mu_{1E_1}x\wedge(\mu_{2E}x)^c$

$=(\mu_{1B_1}x \wedge \mu_{1C_1}x) \wedge ((\mu_{2B} \vee \mu_{2C})x\ )^c$

$=\begin{cases}\mu_{1B_1}x\wedge(\mu_{2B}x)^c = \bar{B}x, for\ x \in B, x \notin C \\ \mu_{1B_1}x\wedge(\mu_{2B}x)^c = \bar{B}x, for\ x \in B \cap C\end{cases}$

Hence $\bar{E}x = \bar{B}x$ for each $x \in E = B$

$\Phi_{\mathcal{A}} \cap \mathcal{C} = \Phi_{\mathcal{A}}$ follows from corollary 1.15.1

Proof (5): we calculate $\bar{D}$ in $\mathcal{B}\cup\mathcal{C}$ from (1c) as follows

$\bar{D}: D \longrightarrow L_D$ is given by, $\bar{D}x = \mu_{1D_1}x\wedge(\mu_{2D}x)^c$

$\bar{D}x = (\mu_{1B_1} \vee \mu_{1C_1})x \wedge (\ \mu_{2B}x \wedge \mu_{2C}x)^c$, for each $x \in D = B \cap C$

$=(\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge [(\mu_{2B}x)^c \vee (\mu_{2C}x)^c]$

$=[(\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge (\mu_{2B}x)^c] \vee [(\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge (\mu_{2C}x)^c]$

$=[\mu_{1B_1}x \wedge (\mu_{2B}x)^c] \vee [\mu_{1C_1}x \wedge (\mu_{2B}x)^c] \vee [\mu_{1B_1}x \wedge (\mu_{2C}x)^c] \vee [\mu_{1C_1}x \wedge (\mu_{2C}x)^c]$

$=\bar{B}x \vee \bar{C}x \vee [\mu_{1C_1}x \wedge (\mu_{2B}x)^c] \vee [\mu_{1B_1}x \wedge (\mu_{2C}x)^c]$

Let $\mathcal{C}\cup\mathcal{B}=\mathcal{F}=(F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$, where

(5a) $F_1 = C_1 \cup B_1$, $F = C \cap B$

(5b) $L_F = L_C \vee L_B$

(5c) $\mu_{1F_1}: E_1 \longrightarrow L_F$ is given by $\mu_{1F_1}x = (\mu_{1C_1} \vee \mu_{1B_1})x$

$\mu_{2F} : F \longrightarrow L_F$ is given by $\mu_{2E}x = \mu_{2C}x \wedge \mu_{2B}x$

$\bar{F}: F \longrightarrow L_F$ is given by $\bar{F}x = \mu_{1F_1}x\wedge(\mu_{2F}x)^c$

$\bar{F}x = (\mu_{1C_1} \vee \mu_{1B_1})x \wedge (\ \mu_{2C}x \wedge \mu_{2B}x)^c$,for each $x \in F = C \cap B$

$=(\mu_{1C_1}x \vee \mu_{1B_1}x) \wedge [(\mu_{2C}x)^c \vee (\mu_{2B}x)^c]$

$=[(\mu_{1C_1}x \vee \mu_{1B_1}x) \wedge (\mu_{2C}x)^c] \vee [(\mu_{1C_1}x \vee \mu_{1B_1}x) \wedge (\mu_{2B}x)^c]$

$=[\mu_{1C_1}x \wedge (\mu_{2C}x)^c] \vee [\mu_{1B_1}x \wedge (\mu_{2C}x)^c] \vee [\mu_{1C_1}x \wedge (\mu_{2B}x)^c] \vee [\mu_{1B_1}x \wedge (\mu_{2B}x)^c]$

$=\bar{C}x \vee \bar{B}x \vee [\mu_{1B_1}x \wedge (\mu_{2C}x)^c] \vee [\mu_{1C_1}x \wedge (\mu_{2B}x)^c]$

Sufficient to show $\mathcal{D}=\mathcal{F}$ i.e.

(5d) $D_1 = F_1, D = F$

(5e) $L_D = L_F$

(5f) $(\mu_{1D_1} = \mu_{1F_1}, \mu_{2D} = \mu_{2F})$ or $\bar{D}x = \bar{F}x$

(5d) follows from (1a) and (5a).

(5e) follows from (1b) and (5b).

(5f) follows from (1c) and (5c).

Proof (6): We calculate $\bar{E}$ in $\mathcal{B}\cap\mathcal{C}$ from (2c) as follows.

$\bar{E}:E \longrightarrow L_E$ is given by, $\bar{E}x = \mu_{1E_1}x\wedge(\mu_{2E}x)^c$

$\bar{E}x = (\mu_{1B_1}x \wedge \mu_{1C_1}x) \wedge ((\mu_{2B} \vee \mu_{2C})x)^C$, for each $x \in E = B \cup C$

$=(\mu_{1B_1}x \wedge \mu_{1C_1}x) \wedge (\mu_{2B}x \vee \mu_{2C}x)^c$

$=(\mu_{1B_1}x \wedge \mu_{1C_1}x) \wedge [(\mu_{2B}x)^c \wedge (\mu_{2C}x)^c]$

$=[\mu_{1B_1}x \wedge (\mu_{2B}x)^c] \wedge [\mu_{1C_1}x \wedge (\mu_{2C}x)^c]$

$=\bar{B}x \wedge \bar{C}x$

Suppose $\mathcal{G}=\mathcal{C}\cap\mathcal{B}=(G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$, where

(6a) $G_1 = C_1 \cap B_1, G = C \cup B$

(6b) $L_G = L_C \wedge L_B$

(6c) $\mu_{1G_1}: G_1 \longrightarrow L_G$ is given by $\mu_{1G_1}x = \mu_{1C_1}x \wedge \mu_{1B_1}x$

$\mu_{2G} : G \longrightarrow L_G$ is given by $\mu_{2G}x = (\mu_{2C} \vee \mu_{2B})x$

$\bar{G}:G \longrightarrow L_G$ is given by $\bar{G}x = \mu_{1G_1}x\wedge(\mu_{2G}x)^c$

$\bar{G}x = (\mu_{1C_1}x \wedge \mu_{1B_1}x) \wedge ((\mu_{2C} \vee \mu_{2B})x)^C$,

for each $x \in G = C \cup B$

$=(\mu_{1C_1}x \wedge \mu_{1B_1}x) \wedge (\mu_{2C}x \vee \mu_{2B}x)^c$

$=(\mu_{1C_1}x \wedge \mu_{1B_1}x) \wedge [(\mu_{2C}x)^c \wedge (\mu_{2B}x)^c]$

$=[\mu_{1C_1}x \wedge (\mu_{2C}x)^c] \wedge [\mu_{1B_1}x \wedge (\mu_{2B}x)^c]$

$=\bar{C}x \wedge \bar{B}x$

Need to show $\mathcal{E}=\mathcal{G}$ i.e. sufficient to show that

(6d) $E_1 = G_1, E = G$

(6e) $L_E = L_G$

and

(6f) $(\mu_{1E_1} = \mu_{1G_1}, \mu_{2E} = \mu_{2G})$ or $\bar{E} = \bar{G}$

(6d) follows from (2a) and (6a).

(6e) follows from (2b) and (6b).

(6f) follows from (2c) and (6c).

The proofs o f (7) a nd (8) follow d irectly from t he definitions of Fs-union and Fs-intersection respectively.

K. Proposition:

For any Fs -subsets $\mathcal{B}$, $\mathcal{C}$and $\mathcal{D}$ of$\mathcal{A} = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$,the following a ssociative laws are true:

(I) $\mathcal{B} \cup (\mathcal{C} \cup \mathcal{D}) = (\mathcal{B} \cup \mathcal{C}) \cup \mathcal{D}$

(II) $\mathcal{B} \cap (\mathcal{C} \cap \mathcal{D}) = (\mathcal{B} \cap \mathcal{C}) \cap \mathcal{D}$, whenever Fs-intersections exist.

Proof (I): Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1},\mu_{2B}), L_B)$,

$\mathcal{C}= (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and

$\mathcal{D}=(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$.

Suppose $\mathcal{C} \cup \mathcal{D} = \mathcal{E}, \mathcal{B} \cup \mathcal{E} = \mathcal{F}, \mathcal{B} \cup \mathcal{C} = \mathcal{G}, \mathcal{G} \cup \mathcal{D} = \mathcal{H}$

Now $\mathcal{E}=\mathcal{C}\cup\mathcal{D}=(E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$, where

(1) $E_1 = C_1 \cup D_1$, $E = C \cap D$

(2) $L_E = L_C \vee L_D$

(3) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is given by$\mu_{1E_1}x = (\mu_{1C_1} \vee \mu_{1D_1})x$

$\mu_{2E} : E \longrightarrow L_E$ is given by$\mu_{2E}x = \mu_{2C}x \wedge \mu_{2D}x$

$\bar{E}: E \longrightarrow L_E$ is given by $\bar{E}x = \mu_{1E_1}x\wedge(\mu_{2E}x)^c$

$\mathcal{F}=\mathcal{B}\cup\mathcal{E}=(F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$, where

(4) $F_1 = B_1 \cup E_1 = B_1 \cup (C_1 \cup D_1), F = B \cap E = B \cap (C \cap D)$

(5) $L_F = L_B \vee L_E = L_B \vee (L_C \vee L_D)$

(6) $\mu_{1F_1}: F_1 \longrightarrow L_F$ is given by $\mu_{1F_1}x = (\mu_{1B_1} \vee \mu_{1E_1})x$

$= (\mu_{1B_1} \vee (\mu_{1C_1} \vee \mu_{1D_1}))x$

$\mu_{2F}: F \longrightarrow L_F$ is given by $\mu_{2F}x = \mu_{2B}x \wedge \mu_{2E}x$

$= \mu_{2B}x \wedge (\mu_{2C}x \wedge \mu_{2D}x\ )$

$\bar{F}: F \longrightarrow L_F$ is given by $\bar{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c$

$\mathcal{G}=\mathcal{B}\cup\mathcal{C}=(G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$, where

(7) $G_1 = B_1 \cup C_1$, $G = B \cap C$

(8) $L_G = L_B \vee L_C$

(9) $\mu_{1G_1}: G_1 \longrightarrow L_G$ is defined by $\mu_{1G_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$

$\mu_{2G} : G \longrightarrow L_G$ is defined by$\mu_{2G}x = \mu_{2B}x \wedge \mu_{2C}x$ and

$\bar{G}: G\longrightarrow L_G$ is defined by $\bar{G}x = \mu_{1G_1}x\wedge(\mu_{2G}x)^c$

$\mathcal{H}=\mathcal{G}\cup\mathcal{D}=(H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H)$, where

(10) $H_1 = G_1 \cup D_1 = (B_1 \cup C_1) \cup D_1, H = G \cap D = (B \cap C)$

(11) $L_H = L_G \vee L_D = (L_B \vee L_C) \vee L_D$

(12) $\mu_{1H_1}: H_1 \longrightarrow L_H$ is defined by $\mu_{1H_1}x = (\mu_{1G_1} \vee \mu_{1D_1})x = ((\mu_{1B_1} \vee \mu_{1C_1}) \vee \mu_{1D_1})x$

$\mu_{2H} : H \longrightarrow L_H$ is defined by $\mu_{2H}x = \mu_{2G}x \wedge \mu_{2D}x = (\mu_{2B}x \wedge \mu_{2C}x) \wedge \square_{2D}x$

$\bar{H}: H \longrightarrow L_H$ is defined by $\bar{H}x = \mu_{1H_1}x \wedge (\mu_{2H}x)^c$

Need to show $\mathcal{F}$=$\mathcal{H}$ i.e. sufficient to show

(13) $F_1 = H_1, F = H$

(14) $L_F = L_H$

(15) $(\mu_{1F_1}x = \mu_{1H_1}x$ and $\mu_{2F}x = \mu_{2H}x)$ or $\bar{F}x = \bar{H}x$

(13) follows from (4)and(10).

(14) follows from (5)and(11).

(15) follows from (6)and(12).

Proof (II): Let $\mathcal{J}$=$\mathcal{C}\cap\mathcal{D}$, $\mathcal{K}$=$\mathcal{B}\cap\mathcal{J}$, $\mathcal{M}$=$\mathcal{B}\cap\mathcal{C}$, $\mathcal{N}$=$\mathcal{M}\cap\mathcal{D}$

Now $\mathcal{J}$=$\mathcal{C}\cap\mathcal{D}$=$(J_1, J, \bar{J}(\mu_{1J_1}, \mu_{2J}), L_j)$, where

(16) $J_1 = C_1 \cap D_1, J = C \cup D$

(17) $L_J = L_C \wedge L_D$

(18) $\mu_{1J_1}: J_1 \longrightarrow L_J$ is given by $\mu_{1J_1}x = \mu_{1C_1}x \wedge \mu_{1D_1}x$

$\mu_{2J} : J \longrightarrow L_J$ is given by $\mu_{2J}x = (\mu_{2C} \vee \mu_{2D})x$

and $\bar{J}$:$J \longrightarrow L_J$ is given by $\bar{J}x = \mu_{1J_1}x \wedge (\mu_{2J}x)^c$

$\mathcal{K}$=$\mathcal{B}\cap\mathcal{J}$=$(K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2k}), L_k)$, where

(19) $K_1 = B_1 \cap J_1 = B_1 \cap (C_1 \cap D_1), F = B \cup J = B \cup (C \cup D)$

(20) $L_K = L_B \wedge L_K$=$L_B \wedge (L_C \wedge L_D)$

(21) $\mu_{1K_1}: K_1 \longrightarrow L_K$ is given by $\mu_{1K_1}x = \mu_{1B_1}x \wedge \mu_{1J_1}x$ $=\mu_{1B_1}x \wedge (\mu_{1C_1}x \wedge \mu_{1D_1}x)$

$\mu_{2K} : K \longrightarrow L_K$ is given by $\mu_{2K}x = (\mu_{2B} \vee \mu_{2J})x$ $=(\mu_{2B} \vee (\mu_{2C} \vee \mu_{2D}))x$ and

$\bar{K}$:$K \longrightarrow L_K$ is given by $\bar{K}x = \mu_{1K_1}x \wedge (\mu_{2K}x)^c$

$\mathcal{M}$=$\mathcal{B}\cap\mathcal{C}$=$(M_1, M, \bar{M}(\mu_{1M_1}, \mu_{2\square}), L_M)$, where

(22) $M_1 = B_1 \cap C_1, M = B \cup C$

(23) $L_M = L_B \wedge L_C$

(24) $\mu_{1M_1}: M_1 \longrightarrow L_M$ is given by $\mu_{1M_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$

$\mu_{2M} : M \longrightarrow L_M$ is given by $\mu_{2M}x = (\mu_{2B} \vee \mu_{2C})x$

and $\bar{M}$:$M \longrightarrow L_M$ is given by $\bar{M}x = \mu_{1M_1}x \wedge (\mu_{2M}x)^c$

$\mathcal{N}$=$\mathcal{M}\cap\mathcal{D}$ =$(N_1, N, \bar{N}(\mu_{1N_1}, \mu_{2N}), L_N)$, where

(25) $N_1 = M_1 \cap D_1 = (B_1 \cap C_1) \cap D_1, J = (B \cup C) \cup D$

(26) $L_N = L_M \wedge L_D$=$(L_B \wedge L_C) \wedge L_D$

(27) $\mu_{1N_1}: N_1 \rightarrow L_N$ is given by $\mu_{1N_1}x = \mu_{1M_1}x \wedge \mu_{1D_1}x$ $=(\mu_{1B_1}x \wedge \mu_{1C_1}x) \wedge \mu_{1D_1}x$

$\mu_{2N} : N \longrightarrow L_N$ is given by $\mu_{2N}x = (\mu_{2M} \vee \mu_{2D})x = ((\mu_{2B} \vee \mu_{2C}) \vee \mu_{2D})x$ and

$\bar{N}$:$N \longrightarrow L_N$ is given by $\bar{N}x = \mu_{1N_1}x \wedge (\mu_{2N}x)^c$

Need to show $\mathcal{K}$=$\mathcal{N}$i.e. sufficient to show

(28) $K_1 = N_1, K = N$

(29) $L_K = L_N$

(30) $(\mu_{1K_1}x = \mu_{1N_1}x$ and $\mu_{2K}x = \mu_{2N}x)$ or $\bar{K}x = \bar{N}x$

(28) follows from (19) and (25).

(29) follows from (20) and (26).

(30) follows from (21) and (27)

L. Arbitrary Fs-unions and arbitrary Fs-intersections:

Given a family $(\mathcal{B}_i)_{i\in I}$ of Fs-subset of $\mathcal{A}$=$(A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$,

where $\mathcal{B}_i = (B_{1i}, B_i, \bar{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$,for any $i \in I$

M. Definition of Fs-union is as follows:

Case (1): For I=Φ, define Fs-union of $(\mathcal{B}_i)_{i\in I}$,denoted by $\bigcup_{i\in I}\mathcal{B}_i$ as $\bigcup_{i\in I}\mathcal{B}_i = \Phi_{\mathcal{A}}$, which is Fs-empty set

Case (2): Define for I≠Φ, Fs-union of $(\mathcal{B}_i)_{i\in I}$ denoted by $\bigcup_{i\in I}\mathcal{B}_i$ as follow

$$\bigcup_{i\in I}\mathcal{B}_i = \mathcal{B} = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B), \text{where}$$

(a) $B_1 = \bigcup_{i\in I} B_{1i}$,$B = \bigcap_{i\in I} B_i$

(b) $L_B = \bigvee_{i\in I} L_{B_i}$ = complete subalgebra generated by $\bigcup L_i (L_i = L_{B_i})$

(c) $\mu_{1B_1}: B_1 \rightarrow L_B$ is defined by $\mu_{1B_1}x = (\bigvee_{i\in I}\mu_{1B_{1i}})x = \bigvee_{i\in I_x}\mu_{1B_{1i}}x$, where $I_x = \{i \in I | x \in B_i\}$

$\mu_{2B}: B \longrightarrow L_B$ is defined by $\mu_{2B}x = (\bigwedge_{i\in I}\mu_{2B_i})x = \bigwedge_{i\in I}\mu_{2B_i}x$

$\bar{B}: B \longrightarrow L_B$ is defined by $\bar{B}x = \mu_{1B_1}x \wedge (\mu_{2B}\, x)^c$

a. *Remark:*

We can easily show that (d) $B_1 \supseteq B$ and $\mu_{1B_1}|B \geq \mu_{2B}$

N. Definition of Fs-intersection:

Case (1): For I=Φ, we define Fs-intersection of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcap_{i\in I}\mathcal{B}_i$ as $\bigcap_{i\in I}\mathcal{B}_i = \mathcal{A}$

Case (2): Suppose

$\bigcap_{i\in I} B_{1i} \supseteq \bigcup_{i\in I} B_i$ and $\bigwedge_{i\in I}\mu_{1B_{1i}}|(\bigcup_{i\in I} B_i) \geq \bigvee_{i\in I}\mu_{2B_i}$

Then, we define Fs-intersection of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcap_{i\in I}\mathcal{B}_i$ as follows

$$\bigcap_{i\in I}\mathcal{B}_i = \mathcal{C} = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$$

(a') $C_1 = \bigcap_{i\in I} B_{1i}$, $C = \bigcup_{i\in I} B_i$

(b') $L_C = \bigwedge_{i\in I} L_{B_i}$

(c') $\mu_{1C_1}: C_1 \longrightarrow L_C$ is defined by $\mu_{1C_1}x = (\bigwedge_{i\in I}\mu_{1B_{1i}})x = \bigwedge_{i\in I}\mu_{1B_{1i}}x$

$\mu_{2C}: C \longrightarrow L_C$ is defined by $\mu_{2C}x = (\bigvee_{i\in I}\mu_{2B_i})x = \bigvee_{i\in I_x}\mu_{2B_i}x$, where $I_x = \{i \in I | \, x \in B_i\}$

$\bar{C}: C \longrightarrow L_C$ is defined by $\bar{C}x = \mu_{1C_1}x \wedge (\mu_{2C}\, x)^c$

Case (3): $\bigcap_{i\in I} B_{1i} \not\supseteq \bigcup_{i\in I} B_i$ or $\bigwedge_{i\in I}\mu_{1B_{1i}}|(\bigcup_{\square\in I} B_i) \not\geq \bigvee_{i\in I}\mu_{2B_i}$

We define

$$\bigcap_{i\in I}\mathcal{B}_i = \Phi_{\mathcal{A}}$$

a. *Lemma:*

For a ny F s-subset $\mathcal{B}$=$(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_A)$ and $\mathcal{B}$⊆ $\mathcal{B}_i = (B_{1i}, B_i, \bar{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$

for each $i \in I$.$\bigcap_{i\in I}\mathcal{B}_i$ exists and $\mathcal{B} \subseteq \bigcap_{i\in I}\mathcal{B}_i$

Proof: $\mathcal{B}$⊆ $\mathcal{B}_i = (B_{1i}, B_i, \bar{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$ for each $i \in I$ implies

(1) $B_1 \subseteq B_{1i}$ and $B \supseteq B_i$

(2) $L_B \leq L_{B_i}$

(3) $\mu_{1B_1}x \leq \mu_{1B_{1i}}x$ for each $x \in B_1$and $\mu_{2B}x \geq \mu_{2B_i}x$ for each $x \in B_i$

All the above statements are true for each $i \in I$

Let $\bigcap_{i\in I}\mathcal{B}_i = \mathcal{D} = (D_1, D, \bar{D}(\mu_{1B_1}, \mu_{2B}), L_B)$, where

(4) $D_1 = \bigcap_{i\in I} B_{1i}$, $D = \bigcup_{i\in I} \square_i$

(5) $L_D = \bigwedge_{i\in I} L_{B_i}$

(6) $\mu_{1D_1}: D_1 \longrightarrow L_D$ is defined by $\mu_{1D_1}x = (\bigwedge_{i\in I}\mu_{1B_{1i}})x = \bigwedge_{i\in I}\mu_{1B_{1i}}x$

$\mu_{2D}: D \longrightarrow L_D$ is defined by $\mu_{2D}x = (\bigvee_{i\in I}\mu_{2B_i})x = \bigvee_{i\in I_x}\mu_{2B_i}x$, where $I_x = \{i \in I | \, x \in B_i\}$

$\bar{D}: D \longrightarrow L_D$ is define by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}\, x)^c$

We needs to show that $D_1 \supseteq D$ and $\mu_{1D_1}x \geq \mu_{2D}x$, for each $x \in D = \bigcup_{i\in I} B_i$

$B_1 \subseteq \bigcap_{i\in I} B_{1i} = D_1$ and $B \supseteq \bigcup_{i\in I} B_i = D$ are follows from (1)

Hence D $\subseteq B \subseteq B_1 \subseteq D_1$ .... (I)

$\mu_{1B_1}x \leq (\bigwedge_{i\in I} \mu_{1B_{1i}})x = \mu_{1D_1}x$, for each $x \in B_1$ and $\mu_{2B}x \geq (\bigvee_{i\in I} \mu_{2B_i})x = \mu_{2D}x$, for each $x \in \bigcup_{i\in I} B_i = D$

Hence $\mu_{2D}x \leq \mu_{2B}x \leq \mu_{1B_1}x \leq \mu_{2B}$,for each $x \in \bigcup_{i\in I} B_i = D$ .... (II)

Hence $\bigcap_{i\in I} \mathcal{B}_i$ exists.

$\mathcal{B} \subseteq \mathcal{D}$ follow from(I),(II)and(5)

Hence $\mathcal{B} \subseteq \bigcap_{i\in I} \mathcal{B}_i$

Let $\mathcal{L}(\mathcal{A})$ be th e c ollection o f F s-subsets o f $\mathcal{A}$. Let$(\mathcal{B}_i)_{i\in I}$ be a ny subfamily o f $\mathcal{L}(\mathcal{A})$,where $\mathcal{B}_i = (B_{1i}, B_i, \bar{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$ for each $i \in I$

O. *Proposition:*

$(\mathcal{L}(\mathcal{A}), \bigcap)$ is $\wedge$-complete lattics.

Proof: Case (1): For I=Φ , $\bigcap_{i\in I} \mathcal{B}_i = \mathcal{A}$ which is the largest element of $\mathcal{L}(\mathcal{A})$

Case (2): F or I ≠Φ, let $(\mathcal{B}_i)_{i\in I}$ be a fa mily o f F s-subsets of $\mathcal{A}$ .So that $\bigcap_{i\in I} \mathcal{B}_i$ does not exist

i.e. $\bigcap_{i\in I} \mathcal{B}_i = \Phi_{\mathcal{A}}$ of f irst ki nd. W e p rove t hat$\Phi_{\mathcal{A}}$ is the greatest lower bound of $(\mathcal{B}_i)_{i\in I}$

Suppose $\mathcal{B} \subseteq \mathcal{A}$ such t hat$\Phi_{\mathcal{A}} \subseteq \mathcal{B} \subseteq \mathcal{B}_i$ for $i \in I$. Then form above lemma $\bigcap_{i\in I} \mathcal{B}_i$ exists which is a contradiction. Hence $\Phi_{\mathcal{A}}$ is greatest lower bound

Case (3): For I≠Φ, let Fs-intersection exist and,$\bigcap_{i\in I} \mathcal{B}_i = \mathcal{D} = (D_1, D, \bar{D}(\mu_{1B_1}, \mu_{2B}), L_B)$

(a') $D_1 = \bigcap_{i\in I} B_{1i}$, $D = \bigcup_{i\in I} B_i$

(b') $L_D = \bigwedge_{i\in I} L_{B_i}$

(c') $\mu_{1D_1}: D_1 \longrightarrow L_D$ is defined by $\mu_{1D_1}x = (\bigwedge_{i\in I} \mu_{1B_{1i}})x = \bigwedge_{i\in I} \mu_{1B_{1i}}\, x$

$\mu_{2D}: D \longrightarrow L_D$ is defined by $\mu_{2D}x = (\bigvee_{i\in I} \mu_{2B_i})x$ $= \bigvee_{i\in I_x} \mu_{2B_i}x$, where $I_x = \{i \in I \mid x \in B_i\}$

$\bar{D}: D \longrightarrow L_D$ is defined by,$\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}\, x)^c$

Existence o f Fs-intersection of given f amily i mply the following

(1) $D_1 = \bigcap_{i\in I} B_{1i} \supseteq \bigcup_{i\in I} B_i = D$

(2) $\bigwedge_{i\in I} \mu_{1B_{1i}}\, x \geq (\bigvee_{i\in I} \mu_{2B_i})x$,for $x \in D$

The p roofs o f t he following r esults a re s ufficient to prove the proposition.

(3) $\bigcap_{i\in I} \mathcal{B}_{1i} \subseteq \mathcal{B}_j$ for each $j \in I$

(4) $\mathcal{B}_j \supseteq \mathcal{E} = (E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$ for each $j \in I$,implies $\mathcal{E} \subseteq \mathcal{D}$

Proof (3): We have the following

(d') $D_1 = \bigcap_{i\in I} B_{1i} \subseteq B_{1j}$ $B_j \subseteq \bigcup_{i\in I} B_i = D$, for each $j \in I$

(e') $L_D = \bigwedge_{i\in I} L_{B_i} \leq L_{B_j}$ for each $j \in I$

(f') $\bigwedge_{i\in I} \mu_{1B_{1i}}\, x \leq \mu_{1B_{1j}}$ for each $x \in D_1$ and $(\bigvee_{i\in I} \mu_{2B_i})x \geq \mu_{2B_{1j}}x$ for each $x \in B_j$ and for each $j \in I$

$\bigcap_{i\in I} B_{1i} \subseteq \mathcal{B}_j$ for each $j \in I$ follow from (d'),(e') and (f')

Proof (4): $\mathcal{E} \subseteq \mathcal{B}_j$ implies

(g') $E_1 \subseteq B_{1j}$, $B_j \subseteq E$

(h') $L_E \leq L_{B_j}$

(i') $\mu_{1E_1}x \leq \mu_{1B_{1j}}x$, for each $x \in E_1$ and $\mu_{2E}x \geq \mu_{2B_j}x$ for each $x \in B_j$

All t hese statement (g')(h') and (i') ar e t rue for each $j \in I$

These in term imply

(5) $E_1 \subseteq \bigcap_{i\in I} B_{1i} = D_1$ and $E \supseteq \bigcup_{i\in I} B_i = D$

(6) $L_E \leq \bigwedge_{i\in I} L_{B_i} = L_D$

(7) $\mu_{1\Box_1}x \leq \bigwedge_{i\in I} \mu_{1B_{1i}}\, x$, for each $x \in E_1$ and $\mu_{2E}x \geq (\bigvee_{i\in I} \mu_{2B_i})x$, for each $x \in B_j$

These in term imply $\mathcal{D}$ is th e g reatest lo wer b ound o f the given family.

$(\mathcal{L}(\mathcal{A}), \bigcap)$ is $\wedge$-complete lattics.

a. *Corollary:*

For any Fs-subset $\mathcal{B}$ of$\mathcal{A}$, the following results are true

(i) $\Phi_{\mathcal{A}} \cup \mathcal{B} = \mathcal{B}$

(ii) $\Phi_{\mathcal{A}} \cap \mathcal{B} = \Phi_{\mathcal{A}}$

Proof: These results follow from case (2) of proposition 1.15.

P. *Proposition:*

$(\mathcal{L}(\mathcal{A}), \bigcup)$ is $\vee$-complete lattics

Proof: Case (I): For I=Φ, $\bigcup_{i\in I} \mathcal{B}_i = \Phi_{\mathcal{A}}$ which is a Fs-empty set acting as the least element of $\mathcal{L}(\mathcal{A})$

Case (II): For I≠Φ,$\bigcup_{i\in I} \mathcal{B}_i =$ $\mathcal{B} = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, where

(1) $B_1 = \bigcup_{i\in I} B_{1i}$,$B = \bigcap_{i\in I} B_i$

(2) $L_B = \bigvee_{i\in I} L_{B_i}$= complete subalgebra generated by $\bigcup_{i\in I} L_{B_i}$

(3) $\mu_{1B_1}: B_1 \to L_B$ is defined by $\mu_{1B_1}x = (\bigvee_{i\in I} \mu_{1B_{1i}})x$ $= \bigvee_{i\in I_x} \mu_{1B_{1i}}x$, where $I_x = \{i \in I \mid x \in B_{1i}\}$

$\mu_{2B}: B \longrightarrow L_B$ is defined by $\mu_{2B}x = (\bigwedge_{i\in I} \mu_{2B_i})x$

$\bar{B}: B \longrightarrow L_B$ is define by $\bar{B}x = \mu_{1B_1}x \wedge (\mu_{2B}\, x)^c$

The p roofs o f t he following r esults a re s ufficient to prove the proposition.

(1) $\mathcal{B}_j \subseteq \bigcup_{i\in I} \mathcal{B}_i$ for each $j \in I$

(2) $\mathcal{B}_j \subseteq \mathcal{C} = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ for each $j \in I$, implies $\mathcal{B} \subseteq \mathcal{C}$

Proof (1): We have the following

(a) $B_{1j} \subseteq \bigcup_{i\in I} B_{1i} = B_1$, $B_j \supseteq \bigcap_{i\in I} B_i = B$, for each $j \in I$

(b) $L_{B_j} \leq \bigvee_{i\in I} L_{B_i} = L_B$, for each $j \in I$

(c) $\mu_{1B_{1j}} \leq (\bigvee_{i\in I} \mu_{1B_{1i}})x$, for each $x \in B_{1j}$and $\mu_{2B_{1j}}x \geq \bigwedge_{i\in I} \mu_{2B_{1i}}\, x$, for each $x \in B$

$\mathcal{B}_j \subseteq \bigcup_{i\in I} \mathcal{B}_i$ follow from (a), (b) and (c)

Proof(2): $\mathcal{B}_j \subseteq \mathcal{C}$ implies

(d) $B_{1j} \subseteq C_1$, $C \subseteq B_j$

(e) $L_{B_j} \leq L_C$

(f) $\mu_{1B_{1j}}x \leq \mu_{1C_1}x$, for each $x \in B_{1j}$ and $\mu_{2B_j}x \geq \mu_{2C}x$ for each $x \in B$

All these statement (d),(e) and (f) are true for each $j \in I$

These in term imply

(3) $\bigcup_{i\in I} B_{1i} = B_1 \subseteq C_1$ and $\bigcap_{i\in I} B_i = B \supseteq C$

(4) $\bigvee_{i\in I} L_{B_i} = L_B \leq L_C$

(5) $(\bigvee_{i\in I} \mu_{1B_{1i}})x \leq \mu_{1C_1}x$, for each $x \in B_1$ and $\bigwedge_{i\in I} \mu_{2B_{1i}}\, x \geq \mu_{2C}x$, for each $x \in C$

These i n t erm i mply $\mathcal{B}$ is t he le ast upper b ound o f t he given family

Hence $(\mathcal{L}(\mathcal{A}), \cup)$ is $\vee$-complete lattics

a. Corollary:

$(\mathcal{L}(\mathcal{A}), \cup, \cap)$ is a complete lattice with $\vee$ and $\wedge$

Proof: This result follows from proposition (1.15) and proposition (1.16)

Q. Proposition:

Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and $\mathcal{D}=(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$. Then $\mathcal{B} \cup (\mathcal{C} \cap \mathcal{D})=(\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D})$ provided $\mathcal{C} \cap \mathcal{D}$ exists.

Proof: Let $\mathcal{C} \cap \mathcal{D} = \mathcal{E}=(\mathrm{E}_1, \mathrm{E}, \bar{\mathrm{E}}(\mu_{1\mathrm{E}_1}, \mu_{2E}), L_E)$, where

(a) $E_1 = C_1 \cap D_1$, $E = C \cup D$

(b) $L_E = L_C \wedge L_D$

(c) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is given by $\mu_{1E_1}x = \mu_{1C_1}x \wedge \mu_{1D_1}x$
$\mu_{2E} : E \longrightarrow \mathrm{L}_E$ is given by $\mu_{2E}x = (\mu_{2C} \vee \mu_{2D})x$
$\bar{E}: E \longrightarrow L_E$ is given by $\bar{E}x = \mu_{1E_1}x = (\mu_{1C_1} \wedge \mu_{1D_1})x \wedge [(\mu_{2C} \vee \mu_{2D})x]^c$

Existence of $\mathcal{C} \cap \mathcal{D}$ implies

(1) $C \cup D \subseteq C_1 \cap D_1$ (2) $(\mu_{1C_1} \wedge \mu_{1D_1})x \geq (\mu_{2C} \vee \mu_{2D})x$, for each $x \in E = C \cup D$

Let $\mathcal{B} \cup \mathcal{E} = \mathcal{F}=(F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$, where

(d) $F_1 = B_1 \cup E_1 = B_1 \cup (C_1 \cap D_1), F = B \cap E = B \cap (C \cup D)$

(e) $L_F = L_B \vee L_E = L_B \vee (L_C \wedge \square_D)$

(f) $\mu_{1F_1}: F_1 \longrightarrow L_F$ is given by $\mu_{1F_1}x = (\mu_{1B_1} \vee \mu_{1E_1})x = [\mu_{1B_1} \vee (\mu_{1C_1} \wedge \mu_{1D_1})]x$
$\mu_{2F}: F \longrightarrow L_F$ is given by $\mu_{2F}x = (\mu_{2B} \wedge \mu_{2E})x = [\mu_{2B} \wedge (\mu_{2C} \vee \mu_{2D})]x$
$\bar{F}: F \longrightarrow L_F$ is given by $\bar{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c = [\mu_{1B_1} \vee (\mu_{1C_1} \wedge \mu_{1D_1})]x \wedge [[\mu_{2B} \wedge (\mu_{2C} \vee \mu_{2D})]x]^c$

To prove the existence of right hand side

Let $\mathcal{B} \cup \mathcal{C} = \mathcal{G}=(\mathrm{G}_1, \mathrm{G}, \bar{\mathrm{G}}(\mu_{1\mathrm{G}_1}, \mu_{2G}), L_G)$, where

(g) $G_1 = B_1 \cup C_1$, G$= B \cap C$

(h) $L_G = L_B \vee L_C$

(i) $\mu_{1G_1}: G_1 \longrightarrow L_G$ is defined by $\mu_{1G_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$
$\mu_{2G}: G \longrightarrow \mathrm{L}_G$ is defined by $\mu_{2G}x = \mu_{2B}x \wedge \mu_{2C}$
and $\bar{G}: G \longrightarrow L_G$ is defined by $\bar{G}x = \mu_{1G_1}x \wedge (\mu_{2G}x)^c = (\mu_{1B_1} \vee \mu_{1C_1})x \wedge [(\mu_{2B} \wedge \mu_{2C})x]^c$

Let $\mathcal{B} \cup \mathcal{D} = \mathcal{H}=(H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H)$, where

(j) $H_1 = B_1 \cup D_1, H = B \cap D$

(k) $L_H = L_B \vee L_D$

(l) $\mu_{1H_1}: H_1 \longrightarrow L_H$ is defined by $\mu_{1H_1}x = (\mu_{1B_1} \vee \mu_{1D_1})x$
$\mu_{2H} : H \longrightarrow L_H$ is defined by $\mu_{2G}x = \mu_{2B}x \wedge \mu_{2C}x$
and $\bar{H}: H \longrightarrow L_H$ is defined by $\bar{H}x = \mu_{1H_1}x \wedge (\mu_{2H}x)^c = (\mu_{1B_1} \vee \mu_{1D_1})x \wedge [(\mu_{2B} \wedge \mu_{2D})x]^c$

Let $(\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D}) = \mathcal{G} \cap \mathcal{H} = \mathcal{K} = (K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K)$

(j) $K_1 = G_1 \cap H_1 = (B_1 \cup C_1) \cap (B_1 \cup D_1) = B_1 \cup (C_1 \cap D_1)$, $K = G \cup H = (B \cap C) \cup (B \cap D) = B \cap (C \cup D)$

(k) $L_K = L_G \wedge L_H = (L_B \vee L_C) \wedge (L_B \vee L_D) = L_B \vee (L_C \wedge L_D)$

(l) $\mu_{1K_1}: K_1 \longrightarrow L_K$ is defined by $\mu_{1K_1}x = (\mu_{1G_1} \wedge \mu_{1H_1})x = [(\mu_{1B_1} \vee \mu_{1C_1}) \wedge (\mu_{1B_1} \vee \mu_{1D_1})]x$
$\mu_{2K} : K \longrightarrow L_K$ is defined by $\mu_{2K}x = (\mu_{2G} \vee \mu_{2H})x = [(\mu_{2B} \wedge \mu_{2C}) \vee (\mu_{2B} \wedge \mu_{2D})]x$ and
$\bar{K}: K \longrightarrow L_K$ is defined by $\bar{K}x = \mu_{1K_1}x \wedge (\mu_{2K}x)^c = [(\mu_{1B_1} \vee \mu_{1C_1}) \wedge (\mu_{1B_1} \vee \mu_{1D_1})]x \wedge [[(\mu_{2B} \wedge \mu_{2C}) \vee (\mu_{2B} \wedge \mu_{2D})]x]^c$

Need to show that (3) $K_1 \supseteq K$ and (4) $\mu_{1K_1}x \geq \mu_{2K}x$ for each $x \in K = B \cap (C \cup D)$

(3) $K = B \cap (C \cup D) \subseteq K_1 = B_1 \cup (C_1 \cap D_1)$ follows from (1)

Case (1): $x \in B$ and $x \in C$ and $x \notin D$, $\mu_{1K_1}x = (\mu_{1\square_1} \vee \mu_{1C_1})x \wedge \mu_{1B_1}x = \mu_{1B_1}x \geq \mu_{2B}x \wedge \mu_{2C}x = \mu_{2K}x$

Case (2): $x \in B$ and $x \notin C$ and $x \in D$, $\mu_{1K_1}x = \mu_{1B_1}x \wedge (\mu_{1B_1} \vee \mu_{1D_1})x = \mu_{1B_1}x \geq \mu_{2B}x \wedge \mu_{2D}x = \mu_{2K}x$

Case (3): $x \in B$ and $x \in C \cap D$, $\mu_{1K_1}x = (\mu_{1B_1}x \vee \mu_{1C_1x}) \wedge (\mu_{1B_1}x \vee \mu_{1D_1x}) = \mu_{1B_1}x \wedge (\mu_{1B_1}x \vee \mu_{1D_1x})$
$\mu_{2K}x = \mu_{2B}x \vee (\mu_{2B}x \wedge \mu_{2D}x)$ i.e $\mu_{1K_1} \geq \mu_{2K}$

Therefore $(\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D})$ exist.

Need to show $\mathcal{F} = \mathcal{K}$ i.e. sufficient to show

(p) $F_1 = K_1, F = K$

(q) $L_F = L_K$

(r) ($\mu_{1F_1}x = \mu_{1K_1}x$ and $\mu_{2F}x = \mu_{2K}x$) or $\bar{F}x = \bar{K}x$

(m) follows from (d)and(m)

(n) follows from (e)and(n)
we have to show

(o) $\mu_{1F_1}x = \mu_{1K_1}x$ and $\mu_{2F}x = \mu_{2K}x$) or $\bar{F}x = \bar{K}x$

Case (4) $x \in B, x \in C, x \in D$
$\bar{K}x = \mu_{1K_1}x \wedge (\mu_{2K}x)^c = [(\mu_{1B_1} \vee \mu_{1C_1}) \wedge (\mu_{1B_1} \vee \mu_{1D_1})]x \wedge [[(\mu_{2B} \wedge \mu_{2C}) \vee (\mu_{2B} \wedge \mu_{2D})]x]^c$
$= [(\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge (\mu_{1B_1}x \vee \mu_{1D_1}x)] \wedge [(\mu_{2B}x \wedge \mu_{2C}x) \vee (\mu_{2B}x \wedge \mu_{2D}x)]^c = [\mu_{1B_1}x \wedge (\mu_{1B_1}x \vee \mu_{1D_1}x)] \wedge [\mu_{2B}x \vee (\mu_{2B}x \wedge \mu_{2D}x)]^c = \bar{F}x$

Case (5) $x \in B \cap C, x \notin B \cap D, x \in D_1$
$\bar{K}x = \mu_{1K_1}x \wedge (\mu_{2K}x)^c = [(\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge (\mu_{1B_1}x \vee \mu_{1D_1}x)] \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c = [\mu_{1B_1}x \wedge (\mu_{1B_1}x \vee \mu_{1D_1}x)] \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c$
$\bar{F}x = [\mu_{1B_1}x \wedge (\mu_{1B_1}x \vee \mu_{1D_1}x)] \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c$

Therefore $\bar{F}x = \bar{K}x$

Case (6) $x \in B \cap C, x \notin B \cap D, x \notin D_1$
$\bar{F}x = \mu_{1B_1}x \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c$
$\mu_{1K_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x \wedge (\mu_{1B_1} \vee \mu_{1D_1})x = (\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge \mu_{1B_1}x = \mu_{1B_1}x$
$\mu_{2K}x = [(\mu_{2B} \wedge \mu_{2C}) \vee (\square_{2B} \wedge \mu_{2D})]x = (\mu_{2B} \wedge \mu_{2C})x = \mu_{2B}x \wedge \mu_{2C}x$

Therefore $\bar{K}x = \mu_{1B_1}x \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c = \bar{F}x$

R. Proposition:

Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and $\mathcal{D}=(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$. Then $\mathcal{B} \cap (\mathcal{C} \cup \mathcal{D})=(\mathcal{B} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{D})$ provided R.H.S exists.

Proof: Let $\mathcal{B} \cap \mathcal{C} = \mathcal{E}=(E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$, where

(a) $E_1 = B_1 \cap C_1$, $E = B \cup C$

(b) $L_E = L_B \wedge L_C$

(c) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is define by $\mu_{1E_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$
$\mu_{2E} : E \longrightarrow \mathrm{L}_E$ is defined by $\mu_{2E}x = (\mu_{2B} \vee \mu_{2C})x$
$\bar{E}: E \longrightarrow L_E$ is defined by $\square x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$

Also we have

(1) $B_1 \cap C_1 \supseteq B \cup C$

(2)$\mu_{1B_1}x \wedge \mu_{1C_1}x \geq (\mu_{2B} \vee \mu_{2C})x \geq \mu_{2B}x$, for each $x \in$ B ∪ C

Let $\mathcal{B}\cap\mathcal{D}$ =$\mathcal{F}$=$(F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$,where

(d) $F_1 = B_1 \cap D_1$, $F = B \cup D$

(e) $L_F = L_B \wedge L_D$

(f) $\mu_{1F_1}: F_1 \longrightarrow L_F$ is defined by $\mu_{1F_1}x = \mu_{1B_1}x \wedge \mu_{1D_1}x$

$\mu_{2F} : F \longrightarrow L_F$ is defined by $\mu_{2F}x = (\mu_{2B} \vee \mu_{2D})x$

$\bar{F}: F \longrightarrow L_F$ is defined by $\bar{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c$

Also we have

(3) $B_1 \cap D_1 \supseteq B \cup D$

(4) $\mu_{1B_1}x \wedge \mu_{1D_1}x \geq (\mu_{2B} \vee \mu_{2D})x \geq \mu_{2B}x$, for each $x \in$ B ∪ D

Let $\mathcal{E}\cup\mathcal{F}$=$\mathcal{G}$=$(G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$, where

(g) $G_1 = E_1 \cup F_1 = (B_1 \cap C_1) \cup (B_1 \cap D_1) = B_1 \cap (C_1 \cup D_1)$,$G = E \cap F = (B \cup C) \cap (B \cup D) = B \cup (C \cap D)$

(h) $L_G = L_E \vee L_F$=$(L_B \wedge L_C) \vee (L_B \wedge L_D) = L_B \wedge (L_C \vee L_D)$

(i) $\mu_{1G_1}: G_1 \longrightarrow L_G$ is given by, $\mu_{1G_1}x = (\mu_{1E_1} \vee \mu_{1F_1})x$

=$[(\mu_{1B_1} \wedge \mu_{1C_1}) \vee (\mu_{1B_1} \wedge \mu_{1D_1})]x$

$\mu_{2G} : G \longrightarrow L_G$ is defined by $\mu_{2G}x = \mu_{2E}x \wedge \mu_{2F}x$

=$[(\mu_{2B} \vee \mu_{2C}) \wedge (\mu_{2B} \vee \mu_{2D})]x$ and

$\bar{G}: G\longrightarrow L_G$ is defined by $\bar{G}x = \mu_{1G_1}x \wedge (\mu_{2G}x)^c$

=$[(\mu_{1B_1} \wedge \mu_{1C_1}) \vee (\mu_{1B_1} \wedge \mu_{1D_1})]x \wedge [[(\mu_{2B} \vee \mu_{2C}) \wedge (\mu_{2B} \vee \mu_{2D})]x]^c$

Let $\mathcal{C}\cup\mathcal{D}$=$\mathcal{H}$=$(H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H)$,where

(j) $H_1 = C_1 \cup D_1, H = C \cap D$

(k) $L_H = L_C \vee L_D$

(l) $\mu_{1H_1}: H_1 \longrightarrow L_H$ is given by $\mu_{1H_1}x = (\mu_{1C_1} \vee \mu_{1D_1})x$ $\mu_{2H} : H \longrightarrow L_H$ is given by $\mu_{2H}x = \mu_{2C}x \wedge \mu_{2D}x$

$\bar{H}: H \longrightarrow L_H$ is given by$\bar{H}x = \mu_{1H_1}x \wedge (\mu_{2H}x)^c$

=$(\mu_{1C_1} \vee \mu_{1D_1})x \wedge [(\mu_{2C} \wedge \mu_{2D})x]^c$

Let $\mathcal{B} \cap (\mathcal{C} \cup \mathcal{D}) = \mathcal{B} \cap \mathcal{H}$=$\mathcal{K}$=$(K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K)$, where

(m) $K_1 = B_1 \cap H_1 = B_1 \cap (C_1 \cup D_1), K = B \cup H = B \cup (C \cap D)$

(n) $L_K = L_B \wedge L_H = L_B \wedge (L_C \vee L_D)$

(o) $\mu_{1K_1}: K_1 \longrightarrow L_K$ is given by $\mu_{1K_1}x = (\mu_{1B_1} \wedge \mu_{1H_1})x = [\mu_{1B_1} \wedge (\mu_{1C_1} \vee \mu_{1D_1})]x$

$\mu_{2K}: K \longrightarrow L_K$ is given by $\mu_{2K}x = (\mu_{2B} \vee \mu_{2H})x = [\mu_{2B} \vee (\mu_{2C} \wedge \mu_{2D})]x$

$\bar{K}: K \longrightarrow L_K$ is given by $\bar{K}x = \mu_{1K_1}x \wedge (\mu_{2K}x)^c$

=$[\mu_{1B_1} \wedge (\mu_{1C_1} \vee \mu_{1D_1})]x \wedge [[\mu_{2B} \vee (\mu_{2C} \wedge \mu_{2D})]x]^c$

Need to show that $\mathcal{B} \cap (\mathcal{C} \cup \mathcal{D})$exists i.e. sufficient to show that

(5)$K \subseteq K_1$

(6)$\mu_{1K_1}x \geq \mu_{2K}x$ for each $x \in K = B \cup (C \cap D)$

(5)follows from(1)and(3)

We have to show (6)

Now $x \in B \cup (C \cap D) \Rightarrow \mu_{1K_1}x = [\mu_{1B_1}x \wedge (\mu_{1C_1} \vee \mu_{1D_1})x]$

Case (1): $x \in B_1, x \in C_1, x \in D_1 \Rightarrow \mu_{1K_1}x = [\mu_{1B_1}x \wedge (\mu_{1C_1}x \vee \mu_{1D_1}x)] = [(\mu_{1B_1}x \wedge \mu_{1C_1}x) \vee (\mu_{1B_1} \wedge \mu_{1D_1})x]$

Case (2): $x \in B, x \notin C \cap D$⇒$\mu_{2K}x = \mu_{2B}x, \mu_{1K_1}x \geq \mu_{2B}x = \mu_{2K}x$

$x \in B, x \in C, x \notin D \Rightarrow \mu_{1K_1}x = (\mu_{1B_1}x \wedge \mu_{1C_1}x) \geq \mu_{2B}x = \mu_{2K}x$

$x \in B, x \notin C, x \in D \Rightarrow \mu_{1K_1}x = (□_{1B_1}x \wedge \mu_{1D_1}x) \geq \mu_{2B}x = \mu_{2K}x$

$x \in B, x \in C \cap D$⇒$\mu_{1K_1}x = (\mu_{1B_1}x \wedge \mu_{1C_1}x) \vee (\mu_{1B_1} \wedge \mu_{1D_1})x \geq \mu_{2B}x = \mu_{2K}x$

Therefore $\mu_{1K_1} \geq \mu_{2K}$

Again need to show $\mathcal{G}$=$\mathcal{K}$ i.e. sufficient to show that

(p) $G_1 = K_1, F = K$

(q) $L_G = L_K$

(r) $(\mu_{1G_1}x = \mu_{1K_1}x$ and $\mu_{2G}x = \mu_{2K}x)$ or $\bar{G}x = \bar{K}x$

(p)follows from(g) and (m)

(q)follows from(h) and (n)

We ha ve t o s how $(\mu_{1G_1}x = \mu_{1K_1}x$ and$\mu_{2G}x = \mu_{2K}x)$ or $\bar{G}x = \bar{K}x$ for each $x \in B \cup (C \cap D)$

Case (1): $x \in B$ *and* $x \in (C \cap D) \Rightarrow \mu_{1G_1}$=$(\mu_{1B_1}x \wedge \mu_{1C_1}x) \vee (\mu_{1B_1}x \wedge \mu_{1D_1}x) = \mu_{1B_1}x \wedge (\mu_{1C_1}x \vee \mu_{1D_1}x)$=$\mu_{1K_1}x$

$\mu_{2G}x$=$(\mu_{2B}x \vee \mu_{2C}x) \wedge (\mu_{2B}x \vee \mu_{2D}x) = [\mu_{2B}x \vee (\mu_{2C}x \wedge \mu_{2D}x)] = \mu_{2K}x$

case (2) : $x \in B$, $x \notin C$ *and* $x \in D \Rightarrow$ $\mu_{1G_1}x$=$\mu_{1B_1}x \wedge \mu_{1D_1}x = \mu_{1K_1}x$

$\mu_{2G}x = \mu_{2B}x \wedge (\mu_{2B}x \vee \mu_{2D}x) = \mu_{2B}x = \mu_{2G}x$

$x \in B$, $x \in C$ *and* $x \notin D \Rightarrow \mu_{1G_1}x$=$\mu_{1B_1}x \wedge \mu_{1C_1}x = \mu_{1K_1}x$

$\mu_{2G}x = (\mu_{2B}x \vee \mu_{2D}x) \wedge \mu_{2B}x = \mu_{2B}x = \mu_{2G}x$

These in term imply$\bar{G}x = \bar{K}x$

Example: 1.18.1

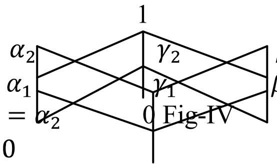


Fig-IV

Let $\mathcal{B}$=$(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$

$B_1 = \{a, b\}, B = \{a\}, L_A = L_B$= α_1

$\mu_{1B_1}: B_1 \longrightarrow L_B$ is given by $\mu_{1B_1} = \alpha_2$

$\mu_{2B}: B \longrightarrow L_B$ is given by $\mu_{2B} = 0$

$\bar{B}: B \longrightarrow L_B$ is given by $\bar{B}x = \alpha_2$

Let $\mathcal{C}$=$(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

$C_1 = \{a, c\}, C = \{a, c\}$, $L_C = L_A$

$\mu_{1C_1}: C_1 \longrightarrow L_C$ is given by $\mu_{1C_1} = \beta_2$

$\mu_{2C}: C \longrightarrow L_C$ is given by $\mu_{2C} = 0$

$\bar{C}: C \longrightarrow L_C$ is given by $\bar{C}x = \beta_2$

Let $\mathcal{D}$=$(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$

$D_1 = \{a, d\}, D = \{a, d\}$,L_D=L_A

$\mu_{1D_1}: D_1 \longrightarrow L_D$ is given by $\mu_{1D_1} = \gamma_2$

$\mu_{2D}: D \longrightarrow L_D$ is given by$\mu_{2D} = 0$

□$: D \longrightarrow L_D$ is given by$\bar{D}x = \gamma_2$

Let $\mathcal{C}\cap\mathcal{D}$ =$\mathcal{E}$=$(E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$,where

(a) $E_1 = C_1 \cap D_1 = \{a\}$, $E = C \cup D$=$\{a, c\}$

(b) $L_E = L_C \wedge L_D$=L_A

(c) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is given by$\mu_{1E_1}x = \mu_{1C_1}x \wedge \mu_{1D_1}x$=$\beta_2 \wedge \gamma_2 = \beta_1$

$\mu_{2E} : E \longrightarrow L_E$ is given by $\mu_{2E}x = (\mu_{2C} \vee \mu_{2D})x = 0$

$\bar{E}: E \longrightarrow L_E$ is given by,$\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$=$\beta_1$

Hence $\mathcal{C}\cap\mathcal{D}$ does not exist.

Let $\mathcal{B}\cup\mathcal{C}$ =$\mathcal{G}$=$(G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$,where

(d) $G_1 = B_1 \cup C_1 = \{a, b, c\}$,G= $B \cap C$=$\{a\}$

(e) $L_G = L_B \vee L_C$=L_A

(f) $\mu_{1G_1}: G_1 \longrightarrow L_G$ is defined by $\mu_{1G_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$=$\alpha_2 \vee \beta_2 = 1$

$\mu_{2G}: G \longrightarrow L_G$ is defined by $\mu_{2G}x = \mu_{2B}x \wedge \mu_{2C}x$ =0

$\bar{G}: G \longrightarrow L_G$ is defined by $\bar{G}x = \mu_{1G_1}x \wedge (\mu_{2G}x)^c = 1$

Let $\mathcal{B} \cup \mathcal{D} = \mathcal{H} = \left(H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H\right)$, where

(g) $H_1 = B_1 \cup D_1 = \{a, b, d\}$, $H = B \cap D = \{a\}$

(h) $L_H = L_B \vee L_D = L_A$

(i) $\mu_{1H_1}: H_1 \longrightarrow L_H$ is defined by $\mu_{1H_1}x = (\mu_{1B_1} \vee \mu_{1D_1})x = \alpha_2 \vee \gamma_2 = 1$

$\mu_{2H}: H \longrightarrow L_H$ is given by $\mu_{2G}x = \mu_{2B}x \wedge \mu_{2C}x = 0$

$\bar{H}: H \longrightarrow L_H$ is given by $\bar{H}x = \mu_{1H_1}x \wedge (\mu_{2H}x)^c = 1$

Let $(\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D}) = \mathcal{G} \cap \mathcal{H} = \mathcal{K} = \left(K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K\right)$

(j) $K_1 = G_1 \cap H_1 = \{a, b\}$, $K = G \cup H = \{a\}$

(k) $L_K = L_G \wedge L_H = L_A$

(l) $\mu_{1K_1}: K_1 \longrightarrow L_K$ is defined by $\mu_{1K_1}x = \left(\mu_{1G_1} \wedge \mu_{1H_1}\right)x = 1$

$\mu_{2K}: K \longrightarrow L_K$ is defined by $\mu_{2K}x = (\mu_{2G} \vee \mu_{2H})x = 0$

$\bar{K}: K \longrightarrow L_K$ is defined by $\bar{K}x = \mu_{1K_1}x \wedge (\mu_{2K}x)^c = 1$

We observed the following

(1) $\mathcal{C} \cap \mathcal{D}$ does not exist i.e. $\mathcal{C} \cap \mathcal{D} = \Phi_A$

(2) L.H.S $\mathcal{B} \cup \Phi_A = \mathcal{B}$

(3) R.H.S $(\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D}) = (\{a, b\}, \{a\}, \bar{K}(1,0), L_A) \neq \mathcal{B}$

Example1.18.2: Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ Fs-subsets of $\mathcal{A}$ as in above example 1.18.3

Let $\mathcal{B} \cap \mathcal{C} = \mathcal{E} = \left(E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E\right)$, where

(1) $E_1 = B_1 \cap C_1 = \{a\}$, $E = B \cup C = \{a, c\}$

(2) $L_E = L_B \wedge L_C = L_A$

(3) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is define by, $\mu_{1E_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x = \alpha_2 \wedge \beta_2 = \gamma_1$

$\mu_{2E}: E \longrightarrow L_E$ is defined by, $\mu_{2E}x = (\mu_{2B} \vee \mu_{2C})x = 0$

$\bar{E}: E \longrightarrow L_E$ is defined by, $\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c = \gamma_1$

$\therefore \mathcal{B} \cap \mathcal{C}$ does not exist

Let $\mathcal{B} \cap \mathcal{D} = \mathcal{F} = \left(F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F\right)$, where

(4) $F_1 = B_1 \cap D_1 = \{a\}$, $F = B \cup D = \{a, c\}$

(5) $L_F = L_B \wedge L_D = L_A$

(6) $\mu_{1F_1}: F_1 \longrightarrow L_F$ is defined by $\mu_{1\square_1}x = \mu_{1B_1}x \wedge \mu_{1D_1}x = \alpha_2 \wedge \gamma_2 = \alpha_1$

$\mu_{2F}: F \longrightarrow L_F$ is defined by $\mu_{2F}x = (\mu_{2B} \vee \mu_{2D})x = 0$

$\bar{F}: F \longrightarrow L_F$ is defined by $\bar{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c = \alpha_1$

$\therefore \mathcal{B} \cap \mathcal{D}$ does not exist

Let $\mathcal{C} \cup \mathcal{D} = \mathcal{H} = \left(H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H\right)$, where

(7) $H_1 = C_1 \cup D_1 = \{a, c, d\}$, $H = C \cap D = \{a\}$

(8) $L_H = L_C \vee L_D = L_A$

(9) $\mu_{1H_1}: H_1 \longrightarrow L_H$ is given by $\mu_{1H_1}x = \left(\mu_{1C_1} \vee \mu_{1D_1}\right)x = \beta_2 \vee \gamma_2 = 1$

$\mu_{2H}: H \longrightarrow L_H$ is given by $\mu_{2H}x = \mu_{2C}x \wedge \mu_{2D}x = 0$

$\bar{H}: H \longrightarrow L_H$ is given by $\bar{H}x = \mu_{1H_1}x \wedge (\mu_{2H}x)^c = 1$

Let $\mathcal{B} \cap (\mathcal{C} \cup \mathcal{D}) = \mathcal{B} \cap \mathcal{H} = \mathcal{K} = \left(K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K\right)$, where

(10) $K_1 = B_1 \cap H_1 = \{a\}$, $K = B \cup H = \{a\}$

(11) $L_K = L_B \wedge L_H = L_A$

(12) $\mu_{1K_1}: K_1 \longrightarrow L_K$ is given by $\mu_{1K_1}x = \left(\mu_{1B_1} \wedge \mu_{1H_1}\right)x = \alpha_2$

$\mu_{2K}: K \longrightarrow L_K$ is given by $\mu_{2K}x = (\mu_{2B} \vee \mu_{2H})x = 0$

$\bar{K}: K \longrightarrow L_K$ is given by $\bar{K}x = \mu_{1K_1}x \wedge (\mu_{2K}x)^c = \alpha_2$

Here R.H.S does not exist.

R.H.S $= \Phi_A$ and L.H.S $= (\{a\}\{a\},, \bar{K}(\alpha_2, 0), L_A)$

III. FS-COMPLEMENTS

A. *Definition of Fs-complement of a Fs-subset:*

Consider a particular Fs-set $\mathcal{A} = \left(A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A\right)$, $A \neq \Phi$, where

(i) $A \subseteq A_1$

(ii) $L_A = [0, M_A]$, $M_A = \vee \bar{A}A = \bigvee_{a \in A} \bar{A}\, a$

(iii) $\mu_{1A_1} = M_A$, $\mu_{2A} = 0$,

$\bar{A}x = \mu_{1A_1}x \wedge (\mu_{2A}x)^c = M_A$, *for each* $x \in A$

Given $\mathcal{B} = \left(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B\right)$. W e d efine Fs-complement of $\mathcal{B}$, denoted by $\mathcal{B}^{C_{\mathcal{A}}}$ for B=A and $L_B = L_A$

$\mathcal{B}^{C_{\mathcal{A}}} = \mathcal{D} = \left(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D\right)$, where

(a') $D_1 = C_A B_1 = B_1^c \cup A$, $D = B = A$

(b') $L_D = L_A$

(c') $\mu_{1D_1}: D_1 \longrightarrow L_A$, is defined by $\mu_{1D_1}x = M_A$

$\mu_{2D}: A \longrightarrow L_A$, is defined by $\mu_{2D}x = \bar{B}x = \mu_{1B_1}x \wedge (\mu_{2B}x)^c$

$\bar{D}: A \longrightarrow L_A$, is defined by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge (\bar{B}x)^c = (\bar{B}x)^c$.

B. *Proposition:* $\mathcal{A}^{C_{\mathcal{A}}} = \Phi_{\mathcal{A}}$

Let $\mathcal{A}^{C_{\mathcal{A}}} = \mathcal{D} = \left(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D\right)$, where

(a') $D_1 = C_A A_1 = A_1^c \cup A = A$, $D = A$

(b') $L_D = L_A$

(c') $\mu_{1D_1}: D_1 \longrightarrow L_A$, is given by $\mu_{1D_1}x = M_A$,

$\mu_{2D}: D \longrightarrow L_A$, is given by $\mu_{2D}x = \bar{A}x$

$\bar{D}: D \longrightarrow L_A$, is given by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge (\bar{A}x)^C = M_A \wedge (M_A)^C = 0$

i.e. $\bar{D} = \bar{0}$, where $\bar{0}x = 0$

Hence $\mathcal{D} = (A, A, \bar{0}(M_A, \bar{A}), L_A) = \Phi_{\mathcal{A}}$, where is an Fs-empty set

C. *Definition: Define* $(\Phi_{\mathcal{A}})^{C_{\mathcal{A}}} = \mathcal{A}$

D. *Proposition: For* $\mathcal{B} = \left(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B\right)$,

$\mathcal{C} = \left(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C\right)$, which are non Fs-empty sets and $B = C = A$, $L_B = L_C = L_A$

(1) $\mathcal{B} \cap \mathcal{B}^{C_{\mathcal{A}}} = \Phi_{\mathcal{A}}$

(2) $\mathcal{B} \cup \mathcal{B}^{C_{\mathcal{A}}} = \mathcal{A}$

(3) $(\mathcal{B}^{C_{\mathcal{A}}})^{C_{\mathcal{A}}} = \mathcal{B}$

(4) $\mathcal{B} \subseteq \mathcal{C}$ if and only if $\mathcal{C}^{C_{\mathcal{A}}} \subseteq \mathcal{B}^{C_{\mathcal{A}}}$

Proof (1): G iven $\mathcal{B} = \left(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B\right)$, wh ere $B = A$, $L_B = L_A$, $\mathcal{B} \neq \Phi_{\mathcal{A}}$

Let $\mathcal{B}^{C_{\mathcal{A}}} = \mathcal{D} = \left(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D\right)$, where

(1a) $D_1 = C_A B_1 = B_1^C \cup A$, $D = B = A$

(1b) $L_D = L_A$

(1c) $\mu_{1D_1}: D_1 \longrightarrow L_A$ is given by $\mu_{1D_1}x = M_A$

$\mu_{2D}: A \longrightarrow L_A$ is given by $\mu_{2D}x = \bar{B}x$,

$\bar{D}: A \longrightarrow L_A$ is given by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge (\bar{B}x)^c = (\bar{B}x)^c$

Let $\mathcal{B} \cap \mathcal{B}^{C_{\mathcal{A}}} = \mathcal{E} = \left(E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E\right)$, where

(1d) $E_1 = B_1 \cap D_1 = B_1 \cap (B_1^C \cup A) = A$ and $E = A$

(1e) $L_E = L_B \wedge L_D = L_A$

(1f) $\mu_{1E_1}x = \mu_{1B_1}x \wedge \mu_{1D_1}x = \mu_{1B_1}x \wedge M_A = \mu_{1B_1}x$, for each $x \in E_1 = A$

$\mu_{2E}x = \mu_{2B}x \vee \mu_{2D}x$, for each $x \in A$

$= \mu_{2B}x \vee \bar{B}x$

$= \mu_{2B}x \vee \left[\mu_{1B_1}x \wedge (\mu_{2B}x)^C\right]$

$=(\mu_{2B}x \vee \mu_{1B_1}x) \wedge [\mu_{2B}x \vee (\mu_{2B}x)^C]$
$= \mu_{1B_1}x \wedge M_A = \mu_{1B_1}x$
$\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c = \mu_{1B_1}x \wedge (\mu_{1B_1}x)^c = 0$, for each $x \in A$
Existence of $\mathcal{B} \cap \mathcal{B}^{C_\mathcal{A}}$ follow from (1d),(1e) and (1f)
Hence $\mathcal{E}=(A, A, \bar{0}(\mu_{1B_1}, \mu_{1B_1}), L_A)$ is an Fs-empty set
$\therefore \mathcal{E}=\mathcal{B} \cap \mathcal{B}^{C_\mathcal{A}}=\Phi_\mathcal{A}$
Proof (2): Let $\mathcal{B} \cup \mathcal{B}^{C_\mathcal{A}} = \mathcal{F} = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$, where
(2a) $F_1 = B_1 \cup D_1 = B_1 \cup (B_1^C \cup A) = A_1, F = B \cap D = A$
(2b) $L_F = L_B \vee L_D = L_A$
(2c) $\mu_{1F_1}x = (\mu_{1B_1} \vee \mu_{1D_1})x$ for each $x \in F_1 = A_1$
$\mu_{2F}x = \mu_{2B}x \wedge \mu_{2D}x$ for each $x \in A$
$= \mu_{2B}x \wedge \bar{B}x$
$= \mu_{2B}x \wedge [\mu_{1B_1}x \wedge (\mu_{2B}x)^C] = 0$
$\bar{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c$, for each $x \in A$
$= (\mu_{1B_1} \vee \mu_{1D_1})x \wedge (0)^c$
$= (\mu_{1B_1}x \vee \mu_{1D_1}x) \wedge M_A$
$= \mu_{1B_1}x \vee \mu_{1D_1}x$
$= \mu_{1B_1}x \vee M_A = M_A$
Hence $\mathcal{F} = (A_1, A, \bar{A}(M_A, 0), L_A) = \mathcal{A}$
$\therefore \mathcal{F}= \mathcal{B} \cup \mathcal{B}^{C_\mathcal{A}} = \mathcal{A}$
Proof (3): Suppose $\mathcal{D}^{C_\mathcal{A}} = \mathcal{G} = (G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$, where
(3a) $G_1 = C_A D_1 = D_1^C \cup A = (B_1^C \cup A)^c \cup A = (B_1 \cap A^c) \cup A = (B_1 \cup A) \cap (A^c \cup A) = B_1 \cap A_1 = B_1, G = D = B = A$
(3b) $L_G = L_B = L_D = L_A$
(3c) $\mu_{1G_1}: G_1 \longrightarrow L_A$, is given by $\mu_{1G_1}x = M_A$
$\mu_{2G}: A \longrightarrow L_A$, is given by $\mu_{2G}x = \bar{D}x = (\bar{B}x)^c$
$\bar{G}: A \longrightarrow L_A$, is given by $\bar{G}x = \mu_{1G_1}x \wedge (\mu_{2G}x)^c = M_A \wedge ((\bar{B}x)^c)^c = \bar{B}x$
We need to show that $\mathcal{B}=\mathcal{G}$
$B_1 = G_1, B = G = A$ follows from (3a)
$L_B = L_G = L_A$ follows from(3b)
$\bar{B}x = \bar{G}x$ follow from(3c)
Hence $\mathcal{B}=\mathcal{G}$
Proof (4): Let $C^{C_\mathcal{A}} = (H_1, H, \bar{H}(\mu_{1H_1}, \mu_{2H}), L_H)$, where
(4a) $H_1 = C_A C_1 = C_1^C \cup A, H = C = A$
(4b) $L_H = L_A$
(4c) $\mu_{1H_1}: H_1 \longrightarrow L_A$, is given by $\mu_{1H_1}x = M_A$
$\mu_{2H}: A \longrightarrow L_A$, is given by $\mu_{2H}x = \bar{C}x$
$\bar{H}: A \longrightarrow L_A$ is given by $\bar{H}x = \mu_{1H_1}x \wedge (\mu_{2H}x)^c = M_A \wedge (\bar{C}x)^c = (\bar{C}x)^c$
($\Longrightarrow$) : Part of the proposition.
Suppose $\mathcal{B} \subseteq \mathcal{C}$, we have the following
(4d) $B_1 \subseteq C_1, C \subseteq B \subseteq A$
(4e) $L_B = L_C = L_A$
(4f) $\mu_{1B_1}x \leq \mu_{1C_1}x$, for each $x \in B_1, \mu_{2B}x \geq \mu_{2C}x$, for each $x \in C$
We need to show $\mathcal{E} \subseteq \mathcal{D}$, that is,
(4g) $E_1 \subseteq D_2, E \supseteq D$
(4h) $L_E \leq L_D$
(4i) $\bar{E}x \leq \bar{D}x$
Therefore
$D_1 = C_A B_1 \supseteq C_A, C_1 = E_1, D = H = A$ follow from(1a)and(4a)
$L_D = L_H = L_A$ follow from(1b)and(4b)
$\mu_{1D_1}x = M_A \geq \mu_{1E_1}x = M_A$, for each $x \in D_1$ and $\mu_{2D}x = (\bar{B}x)^c \geq (\bar{C}x)^c = \mu_{2E}x$, for each $x \in A$
These in term imply $\mathcal{C}^{C_\mathcal{A}} \subseteq \mathcal{B}^{C_\mathcal{A}}$
($\Longleftarrow$) : Part of the proposition.
Let $\mathcal{C}^{C_\mathcal{A}} \subseteq \mathcal{B}^{C_\mathcal{A}}$
From the above result

$$(\mathcal{C}^{C_\mathcal{A}})^{C_\mathcal{A}} \supseteq (\mathcal{B}^{C_\mathcal{A}})^{C_\mathcal{A}} \Rightarrow \mathcal{C} \supseteq \mathcal{B}$$

E. Fs-De-Morgan's laws of a pair of Fs-subset:

For any pair of Fs-sets $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C}= (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$, with $B = C = A$ and $L_B = L_C = L_A$, we will have
(i) $(\mathcal{B} \cup \mathcal{C})^{C_\mathcal{A}} = \mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}}$ if $(\bar{B}x)^c \wedge (\bar{C}x)^c \leq [(\mu_{1B_1}x)^c \vee \mu_{2C}x] \wedge [(\mu_{1C_1}x)^c \vee \mu_{2B}x]$, for each $x \in A$
(ii) $(\mathcal{B} \cap \mathcal{C})^{C_\mathcal{A}} = \mathcal{B}^{C_\mathcal{A}} \cup \mathcal{C}^{C_\mathcal{A}}$, whenever $\mathcal{B} \cap \mathcal{C}$ exist.
Proof (i): First we prove existence of $\mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}}$
Let $\mathcal{B}^{C_\mathcal{A}} = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$, where
(a) $D_1 = C_A B_1 = B_1^C \cup A, D = B = A$
(b) $L_D = L_A$
(c) $\mu_{1D_1}: D_1 \longrightarrow L_A$ given by $\mu_{1D_1}x = M_A$
$\mu_{2D}: A \longrightarrow L_A$ is given by $\mu_{2D}x = \bar{B}x$,
$\bar{D}: A \longrightarrow L_A$ is given by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge (\bar{B}x)^c = (\bar{B}x)^c$
Let $C^{C_\mathcal{A}} = (E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$, where
(d) $E_1 = C_A C_1 = C_1^C \cup A, E = C = A$
(e) $L_E = L_A$
(f) $\mu_{1E_1}: E_1 \longrightarrow L_A$ is defined by $\mu_{1E_1}x = M_A$
$\mu_{2E}: A \longrightarrow L_A$ is defined by $\mu_{2E}x = \bar{C}x$,
$\bar{E}: A \longrightarrow L_A$, is defined by $\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c = M_A \wedge (\bar{c}x)^c = (\bar{C}x)^c$
Let $\mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}} = \mathcal{F} = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$, where
(g) $F_1 = D_1 \cap E_1 = (B_1^c \cup A) \cap (C_1^c \cup A) = (B_1^c \cap C_1^c) \cup A = (B_1 \cup C_1)^c \cup A, \; F = D \cup E = A$
(h) $L_F = L_D \wedge L_E = L_A$
(i) $\mu_{1F_1}x = \mu_{1D_1}x \wedge \mu_{1E_1}x = M_A$, for each $x \in D_1 \cap E_1$
$\mu_{2F}x = \mu_{2D}x \vee \mu_E x = \bar{B}x \vee \bar{C}x$, for each $x \in A$
$\bar{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c = (\bar{B}x \vee \bar{C}x)^c = (\bar{B}x)^c \wedge (\bar{C}x)^c$, for each $x \in A$
$\therefore \mu_{1F_1}x = M_A \geq \bar{B}x \vee \bar{C}x = \mu_{2F}x$
This in term imply existence of $\mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}}$
Case (I): Now we prove the result (i)
Let $\mathcal{G}=\mathcal{B} \cup \mathcal{C}=(G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$, where
(j) $G_1 = B_1 \cup C_1, G = B \cap C$
(k) $L_G = L_B \vee L_C = L_A$
(l) $\mu_{1G_1}: G_1 \longrightarrow L_A$ is given by $\mu_{1G_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$
$\mu_{2G} : G \longrightarrow L_A$ is given by $\mu_{2G}x = \mu_{2B}x \wedge \mu_{2C}x$
$\bar{G}: A \longrightarrow L_A$ is given by $\bar{G}x = \mu_{1G_1}x \wedge (\mu_{2G}x)^c$
$\bar{G}x = (\mu_{1B_1} \vee \mu_{1C_1})x \wedge (\mu_{2B}x \wedge \mu_{2C}x)^c \; \forall x \in G = B \cap C = A$
$=(\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge [(\mu_{2B}x)^c \vee (\mu_{2C}x)^c]$
$=[(\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge (\mu_{2B}x)^c] \vee [(\mu_{1B_1}x \vee \mu_{1C_1}x) \wedge (\mu_{2C}x)^c] = [\mu_{1B_1}x \wedge (\mu_{2B}x)^c] \vee [\mu_{1C_1}x \wedge (\mu_{2B}x)^c]$
$\vee [\mu_{1B_1}x \wedge (\mu_{2C}x)^c] \vee [\mu_{1C_1}x \wedge (\mu_{2C}x)^c]$
$=\bar{B}x \vee \bar{C}x \vee [\mu_{1C_1}x \wedge (\mu_{2B}x)^c] \vee [\mu_{1B_1}x \wedge (\mu_{2C}x)^c]$

$=\bar{B}x \vee \bar{C}x \ (\because \bar{B}x \vee \bar{C}x \geq [\mu_{1C_1}x \wedge (\mu_{2B}x)^c] \vee [\mu_{1B_1}x \wedge (\mu_{2C}x)^c])$

Suppose $\mathcal{H} = (\mathcal{G})^{C_{\mathcal{A}}} = (H_1, H, \bar{H}(\mu_{1\square_1}, \mu_H), L_H)$, where

(m) $H_1 = C_A G_1 = G_1^c \cup A = (B_1 \cup C_1)^c \cup A, H = G = A$

(n) $L_H = L_G = L_A$

(o) $\mu_{1H_1}x = M_A, \forall x \in H_1$, $\mu_{2H}x = \bar{G}x$, for each $x \in A$

$\bar{H}x = M_A \wedge (\bar{G}x)^c = (\bar{G}x)^c = \left[\bar{B}x \vee \bar{C}x \vee [\mu_{1C_1}x \wedge (\mu_{2B}x)^c] \vee [\mu_{1B_1}x \wedge (\mu_{2C}x)^c]\right]^c$

$=(\bar{B}x)^c \wedge (\bar{C}x)^c \wedge [(\mu_{1B_1}x)^c \vee \mu_{2C}x] \wedge [(\mu_{1C_1}x)^c \vee \mu_{2B}x]$

$=(\bar{B}x)^c \wedge (\bar{C}x)^c \ (\because (\bar{B}x)^c \wedge (\bar{C}x)^c \leq [(\mu_{1B_1}x)^c \vee \mu_{2C}x] \wedge [(\mu_{1C_1}x)^c \vee \mu_{2B}x])$

Sufficient to show that $\mathcal{B}^{C_{\mathcal{A}}} \cap (\mathcal{C})^{C_{\mathcal{A}}} = (\mathcal{B} \cup \mathcal{C})^{C_{\mathcal{A}}}$

$F_1 = H_1, F = H = A$ follow from (f) and(m)

$L_F = L_H = L_A$ follow from(g) and (n)

($\mu_{1F_1}x = \mu_{1H_1}x$, for each $x \in F_1, \mu_{2F}x = \mu_{2H}x$, for each $x \in A$) or $\bar{F}x = \bar{H}x$ follow from (h) and (o)

Hence we proved the following

(p) $F_1 = H_1, F = H = A$

(q) $L_F = L_H = L_A$

(r) ($\mu_{1F_1}x = \mu_{1H_1}x$, for each $x \in F_1, \mu_{2F}x = \mu_{2H}x$, for each $x \in A$) or $\bar{F}x = \bar{H}x$

$\mathcal{B}^{C_{\mathcal{A}}} \cap \mathcal{C}^{C_{\mathcal{A}}} = (\mathcal{B} \cup \mathcal{C})^{C_{\mathcal{A}}}$ follow from (p),(q) and(r)

Case (II): If $\mathcal{B}$ is $\Phi_{\mathcal{A}}$ then $\mathcal{B} \cup \mathcal{C} = \Phi_{\mathcal{A}} \cup \mathcal{C} = \mathcal{C}$

$\Rightarrow (\mathcal{B} \cup \mathcal{C})^{C_{\mathcal{A}}} = \mathcal{C}^{C_{\mathcal{A}}}, \mathcal{B}^{C_{\mathcal{A}}} = (\Phi_{\mathcal{A}})^{C_{\mathcal{A}}} = \mathcal{A}$

R.H.S $\mathcal{B}^{C_{\mathcal{A}}} \cap \mathcal{C}^{C_{\mathcal{A}}} = \mathcal{A} \cap \mathcal{C}^{C_{\mathcal{A}}} = \mathcal{C}^{C_{\mathcal{A}}}$

$\therefore (\mathcal{B} \cup \mathcal{C})^{C_{\mathcal{A}}} = \mathcal{B}^{C_{\mathcal{A}}} \cap \mathcal{C}^{C_{\mathcal{A}}}$

Proof (ii): Let $\mathcal{J} = \mathcal{B} \cap \mathcal{C} = (J_1, J, \bar{J}(\mu_{1J_1}, \mu_{2J}), L_J)$, where

(a') $J_1 = B_1 \cap C_1, J = B \cup C$

(b') $L_J = L_B \wedge L_C = L_A$

(c') $\mu_{1J_1}: J_1 \longrightarrow L_A$ is given by $\mu_{1J_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$
$\mu_{2J}: J \longrightarrow L_J$ is given by $\mu_{2J}x = (\mu_{2B} \vee \mu_{2C})x$
and
$\bar{J}: J \longrightarrow L_A$ is given by $\bar{J}x = \mu_{1J_1}x \wedge (\mu_{2J}x)^c$
$\bar{J}x = (\mu_{1B_1}x \wedge \mu_{1C_1}x) \wedge ((\mu_{2B} \vee \mu_{2C})x)^C$,
for each $x \in J = C \cup B = A$
$=(\mu_{1B_1}x \wedge \mu_{1C_1}x) \wedge (\mu_{2B}x \vee \mu_{2C}x)^c$
$=(\mu_{1B_1}x \wedge \mu_{1C_1}x) \wedge [(\mu_{2B}x)^c \wedge (\mu_{2C}x)^c]$
$=[\mu_{1B_1}x \wedge (\mu_{2B}x)^c] \wedge [\mu_{1C_1}x \wedge (\mu_{2C}x)^c]$
$=\bar{B}x \wedge \bar{C}x$

Suppose $\mathcal{N} = \mathcal{J}^{C_{\mathcal{A}}} = (\mathcal{B} \cap \mathcal{C})^{C_{\mathcal{A}}} = (N_1, N, \bar{N}(\mu_{1N_1}, \mu_{2N}), L_N)$, where

(d') $N_1 = C_A J_1 = C_A(B_1 \cap C_1) = (B_1 \cap C_1)^c \cup A, N = J = A$

(e') $L_N = L_J = L_A$

(f') $\mu_{1N_1}x = M_A$, for each $x \in N_1$
$\mu_{2N}x = \bar{J}x$, for each $x \in N = A$
$\bar{N}x = \mu_{1N_1}x \wedge (\mu_{2N}x)^c = M_A \wedge (\bar{J}x)^c = (\bar{J}x)^c$
$=(\bar{B}x \wedge \bar{C}x)^c = (\bar{B}x)^c \vee (\bar{C}x)^c$

Let $\mathcal{B}^{C_{\mathcal{A}}} = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$, where

(g') $D_1 = C_A B_1 = B_1^C \cup A, D = B = A$

(h') $L_D = L_A$

(i') $\mu_{1D_1}: D_1 \longrightarrow L_A$ is given by $\mu_{1D_1}x = M_A$
$\mu_{2D}: A \longrightarrow L_A$ is given by $\mu_{2D}x = \bar{B}x$
$\bar{D}: A \longrightarrow L_A$ is given by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge (\bar{B}x)^c = (\bar{B}x)^c$

Let $\mathcal{C}^{C_{\mathcal{A}}} = (E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$, where

(j') $E_1 = C_A C_1 = C_1^C \cup A, \ E = C = A$

(k') $L_E = L_A$

(l') $\mu_{1E_1}: E_1 \longrightarrow L_A$ is define by $\mu_{1E_1}x = M_A$
$\mu_{2E}: A \longrightarrow L_A$ is define by $\mu_{2E}x = \bar{C}x$
$\bar{E}: A \longrightarrow L_A$,is define by $\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c = M_A \wedge (\bar{C}x)^c = (\bar{C}x)^c$

Let $\mathcal{B}^{C_{\mathcal{A}}} \cup \mathcal{C}^{C_{\mathcal{A}}} = \mathcal{K} = (K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K)$, where

(m') $K_1 = D_1 \cup E_1 = (B_1^c \cup A) \cup (C_1^c \cup A) = (B_1^c \cup C_1^c) \cup A = (B_1 \cap C_1)^c \cup A,$
$K = D \cup E = A$

(n') $L_K = L_D \vee L_E = L_A$

(o') $\mu_{1K_1}x = (\mu_{1D_1} \vee \mu_{1E_1})x$, for each $x \in D_1 \cup E_1$
$\mu_{2K}x = \mu_{2D}x \wedge \mu_{2E}x = \bar{B}x \wedge \bar{C}x$, for each $x \in A$
$\bar{K}x = \mu_{1K_1}x \wedge (\mu_{2K}x)^c$, for each $x \in A$
$=(\mu_{1D_1} \vee \mu_{1E_1})x \wedge (\mu_{2D}x \wedge \mu_{2E}x)^c$
$=(\mu_{1D_1}x \vee \mu_{1E_1}x) \wedge (\bar{B}x \wedge \bar{C}x)^c$
$=(M_A \vee M_A) \wedge [(\bar{B}x)^c \vee (\bar{C}x)^c]$
$=(\bar{B}x)^c \vee (\bar{C}x)^c$

Sufficient to show $\mathcal{N} = \mathcal{K}$

$N_1 = (B_1 \cap C_1)^c \cup A = K_1, N = A = K$ follow from (d') and (m')

$L_K = L_N = L_A$ follow from (e') and (n')

($\mu_{1N_1}x = \mu_{1K_1}x$, for each $x \in N_1, \mu_{2N}x = \mu_{2K}x$, for each $x \in A$) or $\bar{N}x = \bar{K}x$ follow from (f') and (o')

Hence we proved the following

(p') $N_1 \subseteq K_1, \quad N = A = K$

(q') $L_K = L_N = L_A$

(r') ($\mu_{1N_1}x = \mu_{1K_1}x$, for each $x \in N_1, \mu_{2N}x = \mu_{2K}x$, for each $x \in A$)or $\bar{N}x = \bar{K}x$

Hence $(\mathcal{B} \cap \mathcal{C})^{C_{\mathcal{A}}} = \mathcal{B}^{C_{\mathcal{A}}} \cup \mathcal{C}^{C_{\mathcal{A}}}$ follow from (p'),(q')and(r')

a. Example:

There exists a pair of Fs-subset $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ such that

$(\bar{B}x)^c \wedge (\bar{C}x)^c > [(\mu_{1B_1}x)^c \vee \mu_{2C}x] \wedge [(\mu_{1C_1}x)^c \vee \mu_{2B}x]$

and $(\mathcal{B} \cup \mathcal{C})^{C_{\mathcal{A}}} \neq \mathcal{B}^{C_{\mathcal{A}}} \cap \mathcal{C}^{C_{\mathcal{A}}}$

Let $\mathcal{A} = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, where

$A_1 = \{a, b, c\}, A = \{a\}$

$\mu_{1A_1}: A_1 \longrightarrow L_A$ is given by $\mu_{1A_1} = 1$,

$\mu_{2A}: A \longrightarrow L_A$ is given by $\mu_{2A} = 0$,

$\bar{A}: A \longrightarrow L_A$ is given by $\bar{A}x = \mu_{1A_1}x \wedge (\mu_{2A}x)^c = 1 \wedge 0^c = 1$

$\mathcal{B} = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$

$B_1 = \{a, b\}, B = \{a\}$,where $L_A = L_B =$

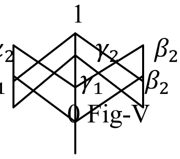


Fig-V

$\mu_{1B_1}: B_1 \longrightarrow L_B$ is given by $\mu_{1B_1} = \alpha_2$

$\mu_{2B}: B \longrightarrow L_B$ is given by $\mu_{2B} = \alpha_1$

$\bar{B}: B \longrightarrow L_B$ is given by

$\bar{B}x = \mu_{1B_1}x \wedge (\mu_{2B}x)^c = \alpha_2 \wedge (\alpha_1)^c = \alpha_2 \wedge \beta_2 = \gamma_1$

$\Rightarrow (\bar{B}x)^c = (\gamma_1)^c = \gamma_2$

$\mathcal{C} = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

$C_1 = \{a, b\}, C = \{a\}$

$\mu_{1C_1}: C_1 \longrightarrow L_C$ is give by $\mu_{1C_1} = \beta_2$

$\mu_{2C}: C \longrightarrow L_C$ is given by $\mu_{2C} = \beta_1$

$\bar{C}: C \longrightarrow L_C$ is given by

$\bar{C}x = \mu_{1C_1}x \wedge (\mu_{2C}x)^c = \beta_2 \wedge (\beta_1)^c = \beta_2 \wedge \alpha_2 = \gamma_1$

$\Rightarrow (\bar{C}x)^c = (\gamma_1)^c = \gamma_2$

Here $\mu_{1B_1}x = \alpha_2 \Rightarrow (\mu_{1B_1}x)^c = (\alpha_2)^c = \beta_1$,

$\mu_{1C_1}x = \beta_2 \Rightarrow (\mu_{1C_1}x)^c = (\beta_2)^c = \alpha_1$

Now $(\bar{B}x)^c \wedge (\bar{C}x)^c = \gamma_2 \wedge \gamma_2 = \gamma_2$ .... (1)

$\left[\left(\mu_{1B_1}x\right)^c \vee \mu_{2C}x\right] \wedge \left[\left(\mu_{1C_1}x\right)^c \vee \mu_{2B}x\right] = (\beta_1 \vee \beta_1) \wedge (\alpha_1 \vee \alpha_1) = \beta_1 \wedge \alpha_1 = 0$ .... (2)

$\therefore (\bar{B}x)^c \wedge (\bar{C}x)^c > \left[\left(\mu_{1B_1}x\right)^c \vee \mu_{2C}x\right] \wedge \left[\left(\mu_{1C_1}x\right)^c \vee \mu_{2B}x\right]$

Let $\mathcal{B}^{C_\mathcal{A}} = \mathcal{D} = \left(D_1, D, \bar{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$, where

(1) $D_1 = C_A B_1 = B_1^C \cup A = \{a, c\}, D = A = \{a\}$

(2) $L_D = L_A$

(3) $\mu_{1D_1}: D_1 \longrightarrow L_A$ is define by $\mu_{1D_1}x = 1$

$\mu_{2D}: A \longrightarrow L_A$ is define by $\mu_{2D}x = \bar{B}x = \gamma_1$

$\bar{D}: A \longrightarrow L_A$ is define by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge (\bar{B}x)^c = (\bar{B}x)^c = \gamma_2$

$\therefore \mathcal{B}^{C_\mathcal{A}} = \mathcal{D} = (\{a, c\}, \{a\}, \bar{D}(1, \gamma_1), L_A)$

Let $\mathcal{C}^{C_\mathcal{A}} = \mathcal{E} = \left(E_1, E, \bar{E}\left(\mu_{1E_1}, \mu_{2E}\right), L_E\right)$, where

(4) $E_1 = C_A C_1 = C_1^C \cup A = \{a, c\}, E = C = A = \{a\}$

(5) $L_E = L_A$

(6) $\mu_{1E_1}: E_1 \longrightarrow L_A$ is given by $\mu_{1E_1}x = 1$

$\mu_{2E}: A \longrightarrow L_A$ is given by $\mu_{2E}x = \bar{C}x = \gamma_1$

and $\bar{E}: A \longrightarrow L_A$ is given by

$\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c = M_A \wedge (\bar{C}x)^c = (\bar{C}x)^c = \gamma_2$

$\therefore \mathcal{C}^{C_\mathcal{A}} = \mathcal{E} = (\{a, c\}, \{a\}, \bar{D}(1, \gamma_1), L_A)$

Let $\mathcal{G} = \mathcal{B} \cup \mathcal{C} = \left(G_1, G, \bar{G}\left(\mu_{1G_1}, \mu_{2G}\right), L_G\right)$, where

(7) $G_1 = B_1 \cup C_1 = \{a, b\}$, $G = B \cap C = \{a\}$

(8) $L_G = L_B \vee L_C = L_A$

(9) $\mu_{1G_1}: G_1 \longrightarrow L_A$ is given by $\mu_{1G_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x = \alpha_2 \vee \beta_2 = 1$

$\mu_{2G}: G \longrightarrow L_A$ is given by $\mu_{2G}x = \mu_{2B}x \wedge \mu_{2C}x = \alpha_1 \wedge \beta_1 = 0$

$\bar{G}: A \longrightarrow L_A$ is given by $\bar{G}x = \mu_{1G_1}x \wedge (\mu_{2G}x)^c = 1 \wedge (0)^c = 1$

Suppose $\mathcal{H} = (\mathcal{G})^{C_\mathcal{A}} = \left(H_1, H, \bar{H}\left(\mu_{1H_1}, \mu_H\right), L_H\right)$, where

(10) $H_1 = C_A G_1 = G_1^c \cup A = (B_1 \cup C_1)^c \cup A = \{a, c\}$, $H = G = A = \{a\}$

(11) $L_H = L_G = L_A$

(12) $\mu_{1H_1}x = 1$, for each $x \in H_1$

$\mu_{2H}x = \bar{G}x = 1$, for each $x \in A$

$\bar{H}x = \mu_{1H_1}x \wedge (\mu_{2H}x)^c = 1 \wedge (1)^c = 0$

$\therefore \mathcal{H} = (\mathcal{G})^{C_\mathcal{A}} = (\{a, c\}, \{a\}, \bar{H}(1,1), L_A)$

Let $\mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}} = \mathcal{F} = \left(F_1, F, \bar{F}\left(\mu_{1F_1}, \mu_{2F}\right), L_F\right)$, where

(13) $F_1 = D_1 \cap E_1 = (B_1^c \cup A) \cap (C_1^c \cup A) = (B_1^c \cap C_1^c) \cup A = (B_1 \cup C_1)^c \cup A = \{a, c\}$, $F = D \cup E = A = \{a\}$

(14) $L_F = L_D \wedge L_E = L_A$

(15) $\mu_{1F_1}x = \mu_{1D_1}x \wedge \mu_{1E_1}x = 1$, for each $x \in D_1 \cap E_1$

$\mu_{2F}x = \mu_{2D}x \vee \mu_E x = \bar{B}x \vee \bar{C}x = \gamma_1 \vee \gamma_1 = \gamma_1$, for each $x \in A$

$\bar{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c = (\bar{B}x \vee \bar{C}x)^c = (\gamma_1)^c = \gamma_2$, for each $x \in A$

$\therefore \mu_{1F_1}x = 1 \geq \bar{B}x \vee \bar{C}x = \gamma_1 = \mu_{2F}x$

This in term imply existence of $\mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}}$ and $\mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}} = \mathcal{F} = (\{a, c\}, \{a\}, \bar{F}(1, \gamma_1), L_A)$

We observed that $\mathcal{H} = (\mathcal{B} \cup \mathcal{C})^{C_\mathcal{A}} \neq \mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}} = \mathcal{G}$

IV. FS-DE MORGAN LAWS OF ANY ARBITRARY FAMILY OF FS-SETS PROPOSITION

Given a family of Fs-subsets $(\mathcal{B}_i)_{i \in I}$ of $\mathcal{A} = \left(A_1, A, \bar{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$, where $L_A = \bigvee_{a \in A} \bar{A}a$, $\mu_{1A_1} = M_A, \mu_{2A} = 0, \bar{A}x = M_A$

(I) $\left(\bigcup_{i \in I} \mathcal{B}_i\right)^{C_\mathcal{A}} = \bigcap_{i \in I} \mathcal{B}_i^{C_\mathcal{A}}$, fo r I≠Φ, w here $\mathcal{B}_i = \left(B_{1i}, B_i, \bar{B}_i\left(\mu_{1B_{1i}}, \mu_{2B_i}\right), L_{B_i}\right)$ and

(1) $B_i = A, L_{B_i} = L_A$ provided $\bigwedge_{i \in I}(\bar{B}_i x)^c \leq \bigwedge_{\substack{i,j \in I \\ i \neq j}} \left[\left(\mu_{1B_{1i}}x\right)^c \vee \mu_{2B_j}x\right]$

(II) $\left(\bigcap_{i \in I} \mathcal{B}_i\right)^{C_\mathcal{A}} = \bigcup_{i \in I} \mathcal{B}_i^{C_\mathcal{A}}$, whenever $\bigcap_{i \in I} \mathcal{B}_i$ exist.

Proof (I): For I=Φ, $\bigcup_{i \in I} \mathcal{B}_i = \Phi_\mathcal{A}$

L.H.S: $(\Phi_\mathcal{A})^{C_\mathcal{A}} = \mathcal{A}$ and R.H.S: $\bigcap_{i \in I} \mathcal{B}_i^{C_\mathcal{A}} = \mathcal{A}$

Hence Fs- De Morgan law holds for I=Φ.

For I≠Φ, first we prove that existence of $\bigcap_{i \in I} \mathcal{B}_i^{C_\mathcal{A}}$

Let $\mathcal{B}_i^{C_\mathcal{A}} = \mathcal{D}_i = \left(D_{1i}, D_i, \bar{D}_i\left(\mu_{1D_{1i}}, \mu_{2D_i}\right), L_{D_i}\right)$, where

(1) $D_{1i} = C_A B_{1i} = B_{1i}^c \cup A, D_i = B_i = A$

(2) $L_{D_i} = L_{B_i} = L_A$

(3) $\mu_{1D_{1i}}x = M_A$, for each $x \in D_{1i}$

$\mu_{2D_i}x = \bar{B}_i x$, for each $x \in D_i = A$

$\bar{D}x = \mu_{1D_{1i}}x \wedge \left(\mu_{2D_i}x\right)^c = M_A \wedge (\bar{B}_i x)^c = (\bar{B}_i x)^c$, for each $x \in D_i = A$

Let $\bigcap_{i \in I} \mathcal{B}_i^{C_\mathcal{A}} = \bigcap_{i \in I} \mathcal{D}_i = \mathcal{D} = \left(D_1, D, \bar{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$, where

(4) $D_1 = \bigcap_{i \in I} D_{1i} = \bigcap_{i \in I}(B_{1i}^c \cup A) = \left(\bigcap_{i \in I} B_{1i}^c\right) \cup A, D = D_i = A$

(5) $L_D = \bigwedge_{i \in I} L_{D_i} = L_A$

(6) $\mu_{1D_1}: D_1 \longrightarrow L_A$ is given by $\mu_{1D_1}x = \bigwedge_{i \in I} \mu_{1D_{1i}}x = M_A$

$\mu_{2D}: D \longrightarrow L_A$ is gi ven by $\mu_{2D}x = \bigvee_{i \in I} \mu_{2D_i}x = \bigvee_{i \in I} \bar{B}_i x$

$\bar{D}x: D \longrightarrow L_A$ is given by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge \left(\bigvee_{i \in I} \bar{B}_i x\right)^c = \left(\bigvee_{i \in I} \bar{B}_i x\right)^c = \bigwedge_{i \in I}(\bar{B}_i x)^c$

$D_1 = \left(\bigcap_{i \in I} B_{1i}\right)^c \cup A \supseteq D = A$ follows from (4)and

$\mu_{1D_1}x = M_A \geq \mu_{2D}x = \bigvee_{i \in I} \bar{B}_i x$ follows from (6)

This shows the existence of $\bigcap_{i \in I} \mathcal{B}_i^{C_\mathcal{A}}$

New $\mathcal{B}_i \subseteq \bigcup_{i \in I} \mathcal{B}_i \Rightarrow (\mathcal{B}_i)^{C_\mathcal{A}} \supseteq \left(\bigcup_{i \in I} \mathcal{B}_i\right)^{C_\mathcal{A}} \Rightarrow \bigcap_{i \in I}(\mathcal{B}_i)^{C_\mathcal{A}} \supseteq \left(\bigcup_{i \in I} \mathcal{B}_i\right)^{C_\mathcal{A}}$ .... (i)

Sufficient to show that $\bigcap_{i \in I}(\mathcal{B}_i)^{C_\mathcal{A}} \subseteq \left(\bigcup_{i \in I} \mathcal{B}_i\right)^{C_\mathcal{A}}$

Let $\bigcup_{i \in I} \mathcal{B}_i = \mathcal{B} = \left(B_1, B, \bar{B}\left(\mu_{1B_1}, \mu_{2B}\right), \square_B\right)$, where

(6) $B_1 = \bigcup_{i \in I} B_{1i}, B = \bigcap_{i \in I} B_i = A$

(7) $L_B = \bigvee_{i \in I} L_{B_i} = L_A$

(8) $\mu_{1B_1}: B_1 \rightarrow L_B$ is given by $\mu_{1B_1}x = \left(\bigvee_{i \in I} \mu_{1B_{1i}}\right)x$

$\mu_{2B}: B \longrightarrow L_B$ is d efine b y $\mu_{2B}x = \left(\bigwedge_{i \in I} \mu_{2B_i}\right)x = \bigwedge_{i \in I} \mu_{2B_i}x$

$\bar{B}: B \longrightarrow L_B$ is define by, $\bar{B}x = \mu_{1B_1}x \wedge (\mu_{2B}\,x)^c$

$= \left(\bigvee_{i \in I} \mu_{1B_{1i}}\right)x \wedge \left(\bigwedge_{i \in I} \mu_{2B_i}x\right)^c$

$= \bigvee_{i \in I} \mu_{1B_{1i}}\,x \wedge \left[\bigvee_{i \in I}\left(\mu_{2B_i}x\right)^c\right]$

Let $\left(\bigcup_{i \in I} \mathcal{B}_i\right)^{C_\mathcal{A}} = E = \left(E_1, E, \bar{E}\left(\mu_{1E_1}, \mu_{2E}\right), L_E\right)$, where

(9) $E_1 = C_A B_1 = C_A \bigcup_{i \in I} B_{1i} = \left(\bigcup_{i \in I} B_{1i}\right)^c \cup A = \left(\bigcap_{i \in I} B_{1i}^c\right) \cup A$, $E = B = A$

(10) $L_E = L_B = L_A$

(11) $\mu_{1E_1}x = M_A$, for each $x \in E_1$

$\mu_{2E}x = \bar{B}x$, for each $x \in A$

$\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$, for each $x \in A$

$= M_A \wedge (\bar{B}x)^c$

$= (\bar{B}x)^c$

$= \left[\bigvee_{i \in I} \mu_{1B_{1i}}\,x \wedge \left[\bigvee_{i \in I}\left(\mu_{2B_{1i}}x\right)^c\right]\right]^c$

$= \bigwedge_{i \in I}\left(\mu_{1B_{1i}}x\right)^c \vee \left[\bigwedge_{i \in I} \mu_{2B_i}x\right]$

$=\wedge_{i\in I}\left[\left(\mu_{1B_{1i}}x\right)^c \vee \mu_{2B_i}x\right] \wedge [\wedge_{i,j\in I, i\neq j}\left(\mu_{1B_{1i}}x\right)^c \vee$
$\mu_{2B_j}x] = \wedge_{i\in I}\left[\mu_{1B_{1i}}x \wedge \left(\mu_{2B_i}x\right)^c\right] \wedge$
$\left[\wedge_{i,j\in I, i\neq j}\left(\mu_{1B_{1i}}x\right)^c \vee \mu_{2B_j}x\right]$
$=\wedge_{i\in I}(\bar{B}_i x)^c = \bar{D}x$

Needs to show $\mathcal{D}\subseteq\mathcal{E}$

(13) $D_1 \subseteq E_1, D \supseteq E$

(14) $L_D \leq L_E$

(15)$\left(\mu_{1D_1}x \leq \mu_{1E_1}x,\right.$ for each $x \in D_1, \mu_{2D}x \geq$ $\mu_{2E}x,$ for each $\left.x \in E\right)$ or $\bar{D}x \leq \bar{E}x$

(13) follow from (4) and (10)

(14) follow from (5) and (11)

(15) follow from (6) and (12)

Hence $\cap_{i\in I}(\mathcal{B}_i)^{C_\mathcal{A}} \subseteq (\cup_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}}$ .... (ii)

Hence $\cap_{i\in I}(\mathcal{B}_i)^{C_\mathcal{A}} = (\cup_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}}$

Proof (II): For I=Φ,$\cap_{i\in I}\mathcal{B}_i = \mathcal{A}$

L.H.S: $(\cap_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}} = (\mathcal{A})^{C_\mathcal{A}} = \Phi_\mathcal{A}$

R.H.S: $\cup_{i\in I}\mathcal{B}_i^{C_\mathcal{A}} = \Phi_\mathcal{A}$

Hence De-Morgan's law holds for I=Φ

For I≠Φ,

Let $\mathcal{B}_i^{C_\mathcal{A}} = \mathcal{D}_i = \left(D_{1i}, D_i, \bar{D}_i\left(\mu_{1D_{1i}}, \mu_{2D_i}\right), L_{D_i}\right)$, where

(1) $D_{1i} = C_A B_{1i} = B_{1i}^c \cup A, D_i = B_i = A$

(2) $L_{D_i} = L_{B_i} = L_A$

(3) $\mu_{1D_{1i}}x = M_A,$ for each $x \in D_{1i}$
$\mu_{2D_i}x = \bar{B}_i x,$ for each $x \in D_i = A$
$\bar{D}x = \mu_{1D_{1i}}x \wedge \left(\mu_{2D_i}x\right)^c = M_A \wedge (\bar{B}_i x)^c =$ $(\bar{B}_i x)^c,$ for each $x \in D_i = A$

Let $\cup_{i\in I}\mathcal{B}_i^{C_\mathcal{A}} = \cup_{i\in I}\mathcal{D}_i = \mathcal{F} = \left(F_1, F, \bar{F}\left(\mu_{1F_1}, \mu_{2F}\right), L_F\right)$, where

(4) $F_1 = \cup_{i\in I} D_{1i} = \cup_{i\in I}(B_{1i}^c \cup A)=(\cup_{i\in I} B_{1i}^c) \cup$ $A, F = \cap_{i\in I} D_i = A$

(5) $L_F = \vee_{i\in I} L_{D_i} = L_A$

(6) $\mu_{1F_1}: F_1 \longrightarrow L_A$ is given by$\mu_{1F_1}x = \left(\vee_{i\in I}\mu_{1D_{1i}}\right)x,$
$\mu_{2F}: F \longrightarrow L_A$ is given by$\mu_{2F}x = \wedge_{i\in I}\mu_{2D_i}x =$ $\wedge_{i\in I}\bar{B}_i x$
$\bar{F}x: F \longrightarrow L_A$is given by $\bar{F}x = \mu_{1F_1}x \wedge (\mu_{2F}x)^c$
$=\left(\vee_{i\in I}\mu_{1D_{1i}}\right)x \wedge \left(\wedge_{i\in I}\mu_{2D_i}x\right)^c = \vee_{i\in I}\mu_{1D_{1i}}x \wedge$ $\left(\wedge_{i\in I}\mu_{2D_i}x\right)^c$ ($\because$ for each $x \in F = A$)
$=M_A \wedge (\wedge_{i\in I}\bar{B}_i x)^c = (\wedge_{i\in I}\bar{B}_i x)^c = \vee_{i\in I}(\bar{B}_i x)^c$

Now

$\cap_{i\in I}\mathcal{B}_i \subseteq \mathcal{B}_i \Rightarrow (\cap_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}} \supseteq {\mathcal{B}_i}^{C_\mathcal{A}} \Rightarrow (\cap_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}} \supseteq$ $\cup_{i\in I}{\mathcal{B}_i}^{C_\mathcal{A}}$.................... (iii)

Sufficient to show $(\cap_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}} \subseteq \cup_{i\in I}{\mathcal{B}_i}^{C_\mathcal{A}}$

$\cap_{i\in I}\mathcal{B}_i = \mathcal{C} = \left(C_1, C, \bar{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$, where

(7) $C_1 = \cap_{i\in I} B_{1i}\,, C = \cup_{i\in I} B_i = A$

(8) $L_i = \wedge_{i\in I} L_{B_i} = L_A$

(9) $\mu_{1C_1}: C_1 \longrightarrow L_C$ is given by, $\mu_{1C_1}x = \left(\wedge_{i\in I}\mu_{1B_{1i}}\right)x$ $= \wedge_{i\in I}\mu_{1B_{1i}}x$
$\mu_{2C}: C \longrightarrow L_C$ is given by$\mu_{2C}x = \vee_{i\in I}\mu_{2B_i}x$
$\bar{C}: C \longrightarrow L_C$ is given by $\bar{C}x = \mu_{1C_1}x \wedge (\mu_{2C}\, x)^c$
$=\wedge_{i\in I}\mu_{1B_{1i}}x \wedge \left(\vee_{i\in I}\mu_{2B_i}x\right)^c$
$=\wedge_{i\in I}\mu_{1B_{1i}}x \wedge \left[\wedge_{i\in I}\left(\mu_{2B_i}x\right)^c\right]$
$=\wedge_{i\in I}\left[\mu_{1B_{1i}}x \wedge \left(\mu_{2B_i}x\right)^c\right]$
$=\wedge_{i\in I}(\bar{B}_i x)$

Let $(\cap_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}} = \mathcal{G} = \left(G_1, G, \bar{G}\left(\mu_{1G_1}, \mu_{2G}\right), L_G\right)$, where

(10) $G_1 = C_A C_1 = C_A(\cap_{i\in I} B_{1i}) = (\cap_{i\in I} B_{1i})^c \cup$ $A = (\cup_{i\in I} B_{1i}^c) \cup A, G = C = A$

(11) $L_G = L_C = L_A$

(12) $\mu_{1G_1}x = M_A,$ for each $x \in G_1$
$\mu_{2G}x = \bar{C}x,$ for each $x \in A$
$\bar{G}x = \mu_{1G_1}x \wedge (\mu_{2G}x)^c,$ for each $x \in A$
$=M_A \wedge (\bar{C}x)^c = (\bar{C}x)^c = (\wedge_{i\in I}(\bar{B}_i x))^c = \vee_{i\in I}(\bar{B}_i x)^c$

Needs to show $\mathcal{G}\subseteq\mathcal{F}$

(13) $G_1 \subseteq F_1, G \supseteq F$

(14) $L_G \leq L_F$

(15) $\left(\mu_{1G_1}x \leq \mu_{1F_1}x,\right.$ for each $x \in G_1, \mu_{2G}x \geq$ $\mu_{2F}x,$ for each $\left.x \in F\right)$ or $\bar{G}x \leq \bar{F}x$

Hence

(13) follow from (4) and (10)

(14) follow from (5) and (11)

(15) follow from (6) and (12)

Hence $\cap_{i\in I}(\mathcal{B}_i)^{C_\mathcal{A}} \subseteq (\cup_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}}$..................(iv)

$\cap_{i\in I}(\mathcal{B}_i)^{C_\mathcal{A}} = (\cup_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}}$ follow from (iii) and (iv)

V. ACKNOWLEDGEMENT

The f irst author de eply acknowledges Nistala V .E.S. Murthy[1] for allowing him to pass through all results of [1] when they were in preparation state and GITAM University Management, Visakhapatnam, A.P-India for providing facilities to do research .

VI. REFERENCES

[1] Nistala V .E.S. M urthy, Is the A xiom o f C hoice T rue for Fuzzy S ets?, J ournal o f Fuzzy Mat hematics, V ol 5(3),P495-523, 1997, U.S.A.

[2] Goguen J .A., L-Fuzzy S ets, Journal o f Mat hematical Analysis and Applications,Vol.18, P145-174,1967

[3] Tridiv J yoti N eog a nd Dusmanta K umar S ut , Complement of an Extended F uzzy S et, International Journal of C omputer A pplications (0975 – 8887),Volume 29– No.3, September 2011

[4] Tridiv J yoti N eog a nd D usmanta K umar Sut, An Extended Approach t o Generalized F uzzy Soft S ets, International J ournal o f E nergy, I nformation a nd Communications Vol. 3, Issue 2, May, 2012

[5] Steven Givant • Paul Halmos, Introduction to Boolean algebras, Springer

[6] Szasz, G ., An I ntroduction to L attice T heory, Academic Press, New York.

[7] Garret B irkhoff, Lattice T heory, American Mathematical S ociety Colloquium p ublications Volume-xxv

[8] Thomas J ech , Set T heory, The Third Millennium Edition revised and expanded, Springer

[9] George J. Klir and Bo Yuan ,Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A. Zadeh ,Advances in Fuzzy Systems-Applications and Theory Vol-6,World Scientific

[10] James D ugundji, T opology, U niversal B ook Stall, Delhi

[11] Nistala V.E.S Murthy and Vaddiparthi Y ogeswara, A Representation T heorem f or Fuzzy S ubsystems o f A

Fuzzy P artial A lgebra, F uzzy S ets a nd S ystem, V ol 104,P359-371,1999,HOLLAND.

[12] Zadeh, L ., Fuzzy S ets, I nformation a nd C ontrol, Vol.8,P338-353,1965

