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GEOMETRIC JACOBIANS DERIVATION AND KINEMATIC SINGULARITY ANALYSIS FOR 6-DOF ROBOTIC MANIPULATOR

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Abstract: This article investigates the kinematic singularities and geometric Jacobians of a 6-DOF robotic manipulator, incorporating a prismatic joint, from the perspective of singularity theory. The study begins by deriving the forward kinematics using the Denavit-Hartenberg (D-H) convention and examines the Jacobian matrices to identify configurations where the Jacobian matrix becomes rank-deficient, signaling the presence of kinematic singularities. These singularities pose critical challenges, such as restricting end-effector mobility and leading to infinite solutions in inverse kinematics. The determinant of the Jacobian matrix is employed to detect singular configurations, and the implications for motion control and trajectory planning are discussed. Through a detailed analysis and MATLAB simulations, the article highlights the importance of singularity avoidance and provides a deeper understanding of the manipulator's kinematic behavior. The findings emphasize the need for strategic design and motion planning to ensure optimal performance and stability in robotic manipulation tasks.

Keywords: Robotic manipulator, kinematic singularities, geometric Jacobian, prismatic joint, Denavit-Hartenberg, inverse kinematics.

I. INTRODUCTION

The kinematics of robotic manipulators are fundamental to their performance in a wide range of applications, including industrial automation, medical robotics, and service robots. A key aspect of this study is the manipulator Jacobian, which serves as the differential kinematic map relating joint velocities to the end-effector's linear and angular velocities. Understanding the behavior of the Jacobian is crucial for trajectory planning, velocity control, force control, and solving the inverse kinematics problem. However, one of the most significant challenges in manipulator kinematics is the occurrence of kinematic singularities, which arise when the Jacobian loses rank, resulting in a loss of degrees of freedom in the system. These singularities limit the motion capabilities of the manipulator and can lead to control issues, such as infinite solutions for inverse kinematics.

This article focuses on a robotic manipulator with six degrees of freedom (6-DOF) and a prismatic joint. Through the derivation of the geometric Jacobian, we explore the impact of joint configurations on the rank of the Jacobian matrix and identify the positions where singularities occur. We employ the Denavit-Hartenberg (D-H) convention for the kinematic modeling of the manipulator and investigate how the Jacobian's determinant can be used to detect singularities. By analyzing the manipulator's kinematics and examining the conditions for singular configurations, this study provides valuable insights into the limitations and potential risks of robotic manipulators, as well as strategies for avoiding these singularities.

II. MANIPULATORS

A manipulator is a type of robotic system comprising a series of mechanical links interconnected by joints, all controlled by a computer. These manipulators form a kinematic chain, with each link connected to the next by a joint. This report specifically investigates rotary joints, which serve as the connection between two adjacent links. The axis of rotation, defined as the intersection of links l_i and l_{i+1}, plays a critical role in the kinematic structure of the manipulator, while the joint variables represent the relative displacement between consecutive links. In this study, we focus on a robotic manipulator with six degrees of freedom, incorporating a prismatic joint within its kinematic configuration.

1 Links and Joint Identification

The manipulator consists of four links, five revolute joints and one prismatic joint. Each joint connects two consecutive links to each other. The links, joints and dimensions are shown in Figure 2.

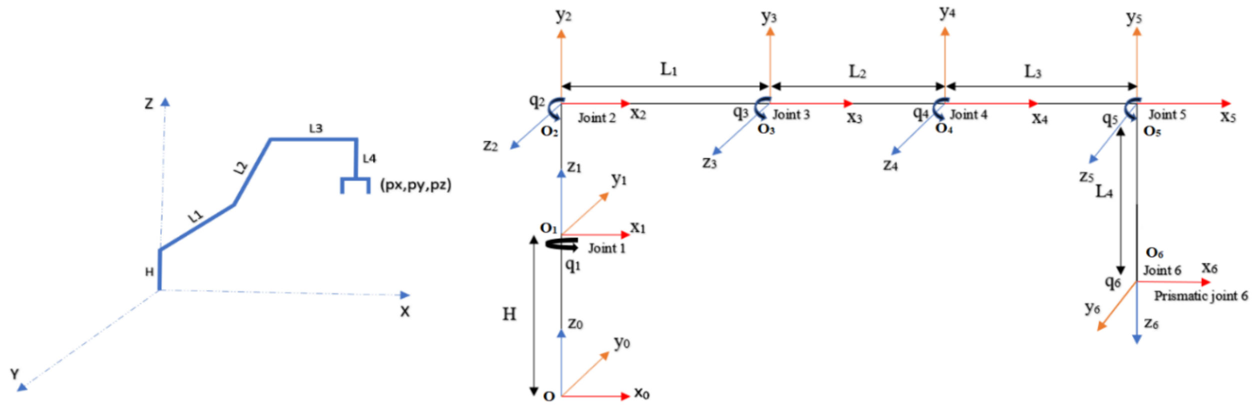

Fig. 2. Links and joint identification

2 D-H parameters

The widely adopted Denavit-Hartenberg (DH) convention utilizes four parameters to define the relationship between the reference frame of each link in a robotic manipulator. These parameters $a_i, d_i, \alpha_i, and\ \theta_i$ are assigned to each link $i \in [1, n]$ to transform the reference frame from link $i-1$ to link i using the fundamental transformation matrices. The coordinate system is established using the D-H convention, with the corresponding parameters outlined in Table 1. For this analysis, the coordinate configuration shown in Figure 2 is selected, and the D-H algorithm is applied to assign the appropriate parameters, as detailed in Table 1.

Table 1 Parameters of the D-H robotic manipulator

Joint i	α_{i-1} (degree)	a_{i-1} (cm)	θ_i (degree)	d_i (cm)
0-1 1 Base	0	0	q_1	H
1-2 2 Shoulder	90	0	q_2	0
2-3 3 Elbow	0	L_1	q_3	0
3-4 4 Elbow	0	L_2	q_4	0
4-5 5 Wrist	0	L_3	q_5	0
5-6 6 Gripper	90	0	q_6	L_4

III. FORWARD KINEMATIC ANALYSIS

Forward kinematics analysis is the process of calculating the position and orientation of the end- effector with given joints angles so by substituting parameters in the homogenous transformation matrix from joint i to joint $i+1$:

$$T_i^{i-1} = \begin{bmatrix} c_{\theta i} & -s_{\theta i}c_{\alpha i} & s_{\theta i}\ s_{\alpha i} & a_i\ c_{\theta i} \\ s_{\theta i} & c_{\theta i}c_{\alpha i} & -c_{\theta i}s_{\alpha i} & a_i\ s_{\theta i} \\ 0 & s_{\alpha i} & c_{\alpha i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

The order of transformations follows a consecutive sequence, beginning from the first joint and continuing to the n^{th} joint. Equation (2) represents the total transformation up to the end-effector.

$$T_i^0 = T_1^0 . T_2^1 . T_3^2 .. T_i^{i-1} = \begin{bmatrix} R_n^0 & P_n^0 \\ 0 & 1 \end{bmatrix} \qquad (2)$$

$$= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^0 = \begin{bmatrix} c_{12}c_{3456} & s_{12} & c_{12}\ s_{3456} & P_x \\ s_{12}\ c_{3456} & -c_{12} & s_{12}\ s_{3456} & P_y \\ s_{3456} & 0 & -c_{3456} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

Where:

$$p_x = L_3\, c_{12}\ c_{345} + L_4\, s_{12} + c_{12}(\, L_2\, s_{34} + L_1\, c_3)$$

$$p_y = L_3\, s_{12}\ c_{345} - L_4\, c_{12} + s_{12}(\, L_2\, c_{34} + L_1\, s_3)$$

$$p_z = H + L_3 s_{345} + L_2 s_3$$

IV. ИНВЕРСНЫЙ КИНЕМАТИЧЕСКИЙ АНАЛИЗ

When working with inverse kinematics, we use the total transformation matrix from equation (2) and multiply it by the inverse of the initial transformation matrix. This allows us to calculate the joint angles by comparing the matrix elements in equation (4).

$$(T_n^0)^{-1} \times T_n^0 = T_2^1 \ldots T_n^{n-1} \qquad (4)$$

where: $(T_n^0)^{-1} = \frac{Adj\,(T_n^0)}{Det\,(T_n^0)}$

To solve q_5 , multiply each side by T_1^{-1} we will get:

$$A_1^{-1} * \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_2^1 * A_3^2 * A_4^3 * A_5^4 * A_6^5 \qquad (5)$$

We can obtain:

$$\begin{bmatrix} c_1 n_x + S_1 n_y & c_1 o_x + S_1 o_y & c_1 a_x + S_1 a_y & c_1 p_x + S_1 p_y \\ -s_1 n_x + C_1 n_y & -s_1 o_x + C_1 o_y & -s_1 a_x + C_1 a_y & -S_1 p_x + S_1 p_y \\ n_z & o_z & a_z & p_z - H \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} . & . & . & L_1 c_2 c_3 + L_2 c_2 s_{34} + L_3 c_2 c_{345} + L_4 s_2 \\ . & . & . & L_1 s_2 c_3 + L_2 s_2 s_{34} + L_3 s_2 c_{345} - L_4 s_2 \\ . & . & . & L_3 s_{345} + L_2 s_{34} + L_1 s_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (6)$$

Both matrix elements in Eq. (6) are equated to each other (right-hand side = and the left-hand side) and the resultant q values are extracted. By taking (1 row, 4 column) (2 row, 4 column):

$$c_1 p_x + s_1 p_y = L_1 c_2 c_3 + L_2 c_2 s_{34} + L_3 c_2 c_{345} + L_4 s_2 \qquad (7)$$

$$-s_1 p_x + c_1 p_y = L_1 s_2 c_3 + L_2 s_2 s_{34} + L_3 s_2 c_{345} - L_4 s_2 \qquad (8)$$

Squaring and adding the two equations, we get:

$$\left(p_x^2 + p_y^2\right) = (L_1 c_3)^2 + L_2^2 + L_3^2 + d_4\, {s_{245}}^2 + 2 * L_1 L_2 c_3 c_4 + 2 * L_1 L_3 c_{45} c_3 +$$

$$2 * L_2 L_3 c_4 c_{45} - 2 * L_1\, d_4\, c_3\, s_{245} - 2 *\, d_4\, L_2\, s_4 c_{245} - 2 *\, d_4\, L_3\, s_{45} c_{245}$$

We can solve for $(\cos q_5, \sin q_5)$ as follows:

$$c_5 = \frac{p_x^2 + p_y^2 - (L_1 c_3)^2 - L_2^2 - L_3^2 - (L_4\, s_{245})^2}{2 * L_2\, L_3}$$

$$s_5 = \mp\sqrt{1 - c_5^2} = \mp\sqrt{1 - \frac{(p_x^2 + p_y^2 - (L_1 c_3)^2 - L_2^2 - L_3^2 - L_4\, s_{245})^2}{(2 * L_2\, L_3)^2}}$$

$$q_5 = \mathrm{Atan2}(S_5, C_5) \qquad (9)$$

From Eq. (6) we can get:

$$p_z - H = L_3 s_{345} + L_2 s_{34} + L_1 s_3 \qquad (10)$$

$$s_{34} = \frac{p_z - H - L_3 s_{345} - L_1 s_3}{L_2} \qquad (11)$$

$$q_{34} = \text{Atan2}\left[\frac{p_z - H - L_3 s_{345} - L_1 s_3}{L_2}, \mp\sqrt{1 - \left(\frac{p_z - H - L_3 s_{345} - L_1 s_3}{L_2}\right)^2}\right] \quad (12)$$

$$q_4 = q_{34} - q_3$$

Multiplying each side of Eq. (2) with $T_1^{-1} * T_2^{-1}$

$$A_1^{-1} * A_2^{-1} * \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_3^2 * A_4^3 * A_5^4 * A_6^5 \quad (13)$$

We can obtain:

$$\begin{bmatrix} c_1c_2\, n_x + c_1s_2\, n_y + s_1n_z & c_1c_2\, o_x + c_1s_2\, o_y + s_1o_z & c_1c_2\, a_x + c_1s_2\, a_y + s_1a_z & c_1c_2\, p_x + c_1\, s_2\, p_y + s_1\, p_z \\ -s_1c_2\, n_x - s_1s_2\, n_y + c_1n_z & -s_1c_2\, o_x - s_1s_2\, o_y + c_1o_z & -s_1c_2\, a_x - s_1s_2\, a_y + c_1a_z & -s_1c_2\, p_x - s_1\, s_2\, p_y + c_1\, p_z \\ s_2\, n_x - c_2\, n_y & s_2\, o_x - c_2\, o_y & s_2\, a_x - c_2\, a_y & s_2\, p_x - c_2\, p_y - H \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{3456} & 0 & s_{3456} & L_3\, c_{345} + L_2\, c_{34} + L_1c_3 \\ s_{3456} & 0 & -c_{3456} & L_3\, s_{345} + L_2\, s_{34} + L_1s_3 \\ 0 & 1 & 0 & L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (14)$$

Equating elements (3,4) of the right-hand side matrix and the left-hand side matrix of Eq. (14), we can obtain:

$$s_2\, p_x - c_2\, p_y - H = L_4$$

$$s_2\, p_x - c_2\, p_y = L_4 + H$$

$$q_2 = \text{Atan2}(p_x, -p_y) \mp \left[\sqrt{{p_x}^2 + {p_y}^2 - (L_4 + H)^2}, (L_4 + H)\right] \quad (15)$$

From Eq. (2) we can obtain:

$$a_x = c_{12}\, s_{3456}$$

$$a_y = s_{12}\, s_{3456}$$

Dividing the two equations:

$$\frac{s_{56}}{c_{56}} = \frac{a_y}{a_x} q_{12} = \text{Atan2}(a_y, a_x) \quad (16)$$

$$q_1 = q_{12} - q_2 \quad (17)$$

Now multiply each side of Eq. (2) by $A_1^{-1} * A_2^{-1} * A_3^{-1}$:

$$A_1^{-1} * A_2^{-1} * A_3^{-1} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_4^3 * A_5^4 * A_6^5 \quad (18)$$

$$\begin{bmatrix} c_1c_{23} & c_1s_{23} & s_1 & -L_1c_1c_2 \\ -s_1c_{23} & -s_1s_{23} & c_1 & L_1s_1s_2 \\ s_{23} & -c_{23} & 0 & -L_1s_2 - H \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_{456} & 0 & s_{456} & L_3c_{45} + L_2c_4 \\ c_{456} & 0 & -c_{456} & L_3s_{45} + L_2s_4 \\ 0 & 1 & 0 & L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (19)$$

Equating elements (3, 4) from the two sides of Eq. (19):

$$s_{23}\, p_x - c_{23} p_y - L_1 s_2 - H = L_4$$

$$s_{23}\, p_x - c_{23} p_y = L_1 s_2 + H + L_4$$

$$q_{23} = \text{Atan2}(p_x, -p_y) \mp \text{Atan2}\left(\sqrt{p_x^2 + p_y^2 - (L_1 s_2 + H + L_4)^2}, L_1 s_2\, H + L_4\right) \quad (20)$$

$$q_3 = q_{23} - q_2 \quad (21)$$

From the Eq. (2) we can also obtain:

$$a_z = -c_{3456}$$

$$C_{1234} = -a_z\, q_{3456} = \text{Atan2}\left(\sqrt{1 - a_z{}^2}, a_z\right) \quad (22)$$

$$q_6 = q_{3456} - q_3 - q_4 - q_5 \quad (23)$$

V. Linear and Angular Velocity

In the context of robotic manipulator kinematics, velocity refers to the rate at which an object or a point on the object is changing its position. In robotic manipulators, linear and angular velocity are important concepts in the field of robotics, particularly when dealing with robot manipulators.

- Linear velocity refers to the rate at which an object moves along a straight path. In the context of a robot manipulator, it represents the speed at which a point on the robot moves in a straight line. In a robot manipulator, the linear velocity of a point on the end-effector is crucial for tasks involving movements in a straight line, such as reaching a specific position in the workspace.
- Angular velocity is significant for tasks involving rotational movements, such as rotating an object. It is also crucial for controlling the orientation of the end-effector. The angular velocity of each joint in a robot manipulator affects the overall motion and orientation of the end-effector.

Linear velocity and angular velocity calculation:

For a revolute joint, the linear velocity and angular velocity calculations can be expressed using the DH parameters and the basic principles of kinematics.

$$v_{i+1}^{i+1} = R_i^{i+1} . (v_i^i + \omega_i^i \times p_{i+1}^i) \quad (1)$$

$$\omega_{i+1}^{i+1} = R_i^{i+1} . \omega_i^i + \dot{q}_{i+1} . z_{i+1}^{i+1} \quad (2)$$

For a prismatic joint, the linear velocity and angular velocity calculations are a simpler compared to a revolute joint because prismatic joints involve translational motion instead of rotational motion.

$$v_{i+1}^i = v_i^i + \omega_i^i \times p_{i+1}^i \quad (3)$$

$$\omega_{i+1}^{i+1} = R_i^{i+1} . \omega_i^i \quad (4)$$

where: R_i^{i+1} relative rotation matrix representing the orientation of frame $i+1$ with respect to frame i. z_{i+1}^{i+1} unit vector along the rotation axis of joint $i+1$ expressed in its own coordinate frame. $\dot{q}_{i+1}$ joint velocity of joint $i+1$. ω_{i+1}^{i+1} represents the angular velocity of the $i+1$ frame.

- ***In joint 1 (Revolute joint q_1):***

$$T_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & H \\ 0 & 0 & 0 & 1 \end{bmatrix}, v_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};\ \omega_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_1^1\ (Angular\ velocity) = R_0^1 . \omega_0^0 + \dot{q_1} . z_1^1 = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{q_1} \end{bmatrix} \quad (5)$$

$$v_1^1 (Linear\ velocity) = R_0^1 . (v_0^0 + \omega_0^0 \times p_1^0) = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ H \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

- ***Joint 2 (Revolute joint q_2)***

$$T_2^1 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \omega_1^1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q_1} \end{bmatrix};\ v_1^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_2^2(Angular\ velocity) = R_1^2 . \omega_1^1 + \dot{q_2} . z_2^2 = \begin{bmatrix} c_2 & s_2 & 0 \\ 0 & 0 & 1 \\ s_2 & -c_2 & 0 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ \dot{q_1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{q_1} \\ \dot{q_2} \end{bmatrix} \quad (7)$$

$$v_2^2(Linear\ velocity) = R_1^2 . (v_1^1 + \omega_1^1 \times p_2^1) = \begin{bmatrix} c_2 & s_2 & 0 \\ 0 & 0 & 1 \\ s_2 & -c_2 & 0 \end{bmatrix} . \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q_1} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

- ***Joint 3 (Revolute joint q_3)***

$$T_3^2 = \begin{bmatrix} c_3 & -s_3 & 0 & c_3.L_1 \\ s_3 & c_3 & 0 & s_3.L_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \omega_2^2 = \begin{bmatrix} 0 \\ \dot{q_1} \\ \dot{q_2} \end{bmatrix};\ v_2^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_3^3(Angular\ velocity) = R_2^3 . \omega_2^2 + \dot{q_3} . z_3^3 = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ \dot{q_1} \\ \dot{q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q_3} \end{bmatrix} = \begin{bmatrix} s_3\dot{q_1} \\ c_3\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} \end{bmatrix} \quad (9)$$

$$v_3^3 = R_2^3 . (v_2^2 + \omega_2^2 \times p_1^2) = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{q_1} \\ \dot{q_2} \end{bmatrix} \times \begin{bmatrix} c_3.L_1 \\ s_3.L_1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -L_1c_3\ s_3\ \dot{q}_2 + L_1c_3\ s_3\ \dot{q}_2 \\ L_1s_3\ s_3\ \dot{q}_2 + L_1c_3\ c_3\ \dot{q}_2 \\ -L_1\ c_3\ \dot{q}_1 \end{bmatrix} \quad (10)$$

- ***Joint 4 (Revolute joint q_4)***

$$T_4^3 = \begin{bmatrix} c_4 & -s_4 & 0 & c_4.L_2 \\ s_4 & c_4 & 0 & s_4.L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \omega_3^3 = \begin{bmatrix} s_3\dot{q_1} \\ c_3\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} \end{bmatrix};\ v_3^3 = \begin{bmatrix} 0 \\ L_1s_3\ s_3\ \dot{q}_2 + L_1c_3\ c_3\ \dot{q}_2 \\ -L_1\ c_3\ \dot{q}_1 \end{bmatrix}$$

$$\boldsymbol{\omega_4^4}(Angular\ velocity) = R_3^4 . \omega_3^3 + \dot{q_4} . z_4^4 = \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_3\dot{q_1} \\ c_3\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q_4} \end{bmatrix} = \begin{bmatrix} s_{34}\dot{q_1} \\ c_{34}\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} + \dot{q_4} \end{bmatrix} \quad (11)$$

$$\boldsymbol{v_4^4} = R_3^4 . (v_3^3 + \omega_3^3 \times p_2^3) = \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \left(\begin{bmatrix} 0 \\ L_1s_3\ s_3\ \dot{q}_2 + L_1c_3\ c_3\ \dot{q}_2 \\ -L_1\ c_3\ \dot{q}_1 \end{bmatrix} + \begin{bmatrix} s_3\dot{q_1} \\ c_3\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} \end{bmatrix} \times \begin{bmatrix} c_4.L_2 \\ s_4.L_2 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -L_2\ s_4\ \dot{q}_2 - L_2\ s_4\ \dot{q}_3 \\ L_1\ s_3\ s_3\ \dot{q}_2 + L_1\ c_3\ c_3\ \dot{q}_2 + L_2\ c_4\ \dot{q}_2 + L_2\ c_4\ \dot{q}_3 \\ -L_1\ c_3\ \dot{q}_1 - L_2\ c_{34}\ \dot{q}_1 \end{bmatrix} \quad (12)$$

- ***Joint 5 (Revolute joint q_5)***

$$T_6^5 = \begin{bmatrix} c_5 & -s_5 & 0 & c_5.L_3 \\ s_5 & c_5 & 0 & s_5.L_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \omega_4^4 = \begin{bmatrix} s_{34}\dot{q_1} \\ c_{34}\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} + \dot{q_4} \end{bmatrix};\ v_4^4 = \begin{bmatrix} -L_2\ s_4\ \dot{q}_2 - L_2\ s_4\ \dot{q}_3 \\ L_1\ s_3\ s_3\ \dot{q}_2 + L_1\ c_3\ c_3\ \dot{q}_2 + L_2\ c_4\ \dot{q}_2 + L_2\ c_4\ \dot{q}_3 \\ -L_1\ c_3\ \dot{q}_1 - L_2\ c_{34}\ \dot{q}_1 \end{bmatrix}$$

$$\boldsymbol{\omega_5^5} = R_4^5 . \omega_4^4 + \dot{q_5} . z_4^4 = \begin{bmatrix} c_5 & s_5 & 0 \\ -s_5 & c_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_{34}\dot{q_1} \\ c_{34}\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} + \dot{q_4} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q_5} \end{bmatrix} = \begin{bmatrix} s_{345}\dot{q_1} \\ c_{345}\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} + \dot{q_4} + \dot{q_5} \end{bmatrix} \quad (13)$$

$$\boldsymbol{v_5^5}(Linear\ velocity) = R_4^5 . (v_4^4 + \omega_4^4 \times p_3^4)$$

$$= \begin{bmatrix} c_5 & s_5 & 0 \\ -s_5 & c_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \left(\begin{bmatrix} -L_2\ s_4\ \dot{q}_2 - L_2\ s_4\ \dot{q}_3 \\ L_1\ s_3\ s_3\ \dot{q}_2 + L_1\ c_3\ c_3\ \dot{q}_2 + L_2\ c_4\ \dot{q}_2 + L_2\ c_4\ \dot{q}_3 \\ -L_1\ c_3\ \dot{q}_1 - L_2\ c_{34}\ \dot{q}_1 \end{bmatrix} + \begin{bmatrix} s_{34}\dot{q_1} \\ c_{34}\ \dot{q_1} \\ \dot{q_2} + \dot{q_3} + \dot{q_4} \end{bmatrix} \times \begin{bmatrix} c_5.L_3 \\ s_5.L_3 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -\mathrm{L}_2\, c_5\, s_4\, \dot{q}_2 - \mathrm{L}_2\, s_4\, c_5\, \dot{q_3} - \mathrm{L}_3 c_5\, s_5(\dot{q}_2 + \dot{q}_3 + \dot{q}_4) + s_5(\mathrm{L}_1\, s_3\, s_3\, \dot{q}_2 + \mathrm{L}_1\, c_3\, c_3\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_3) \\ -s_5(-\mathrm{L}_2\, s_4\, \dot{q}_2 - \mathrm{L}_2\, s_4\, \dot{q}_3) + c_5\,(\mathrm{L}_1\, s_3\, s_3\, \dot{q}_2 + \mathrm{L}_1\, c_3\, c_3\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q_3}) \\ -\mathrm{L}_1\, c_3\, \dot{q}_1 - \mathrm{L}_2\, c_{34}\, \dot{q}_1 \end{bmatrix} \quad (14)$$

- ***Joint 6 (prismatic joint q_6)***

$$A_6^5 = \mathrm{A}_6 = \begin{bmatrix} c_6 & 0 & s_6 & 0 \\ s_6 & 0 & -c_6 & 0 \\ 0 & 1 & 0 & L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \omega_5^5 = \begin{bmatrix} s_{345}\dot{q}_1 \\ c_{345}\,\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 + \dot{q}_4 + \dot{q}_5 \end{bmatrix}$$

$$\mathbf{v_5^5} = \begin{bmatrix} -\mathrm{L}_2\, c_5\, s_4\, \dot{q}_2 - \mathrm{L}_2\, s_4\, c_5\, \dot{q_3} - \mathrm{L}_3 c_5\, s_5(\dot{q}_2 + \dot{q}_3 + \dot{q}_4) + s_5(\mathrm{L}_1\, s_3\, s_3\, \dot{q}_2 + \mathrm{L}_1\, c_3\, c_3\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_3) \\ -s_5(-\mathrm{L}_2\, s_4\, \dot{q}_2 - \mathrm{L}_2\, s_4\, \dot{q}_3) + c_5\,(\mathrm{L}_1\, s_3\, s_3\, \dot{q}_2 + \mathrm{L}_1\, c_3\, c_3\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q_3}) \\ -\mathrm{L}_1\, c_3\, \dot{q}_1 - \mathrm{L}_2\, c_{34}\, \dot{q}_1 \end{bmatrix}$$

$$\boldsymbol{\omega_6^6} = R_5^6\,.\,\omega_5^5 = \begin{bmatrix} c_6 & s_6 & 0 \\ 0 & 0 & 1 \\ -s_6 & c_6 & 0 \end{bmatrix} . \begin{bmatrix} s_{345}\dot{q}_1 \\ c_{345}\,\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 + \dot{q}_4 + \dot{q}_5 \end{bmatrix} = \begin{bmatrix} s_{3456}\,\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 + \dot{q}_4 + \dot{q}_5 \\ c_{3456}\,\dot{q}_1 \end{bmatrix} \quad (15)$$

$$\boldsymbol{v_6^6}(Linear\ velocity) = v_5^5 + \omega_5^5 \times p_6^5$$

$$= \begin{bmatrix} -\mathrm{L}_2\, c_5\, s_4\, \dot{q}_2 - \mathrm{L}_2\, s_4\, c_5\, \dot{q_3} - \mathrm{L}_3 c_5\, s_5(\dot{q}_2 + \dot{q}_3 + \dot{q}_4) + s_5(\mathrm{L}_1\, s_3\, s_3\, \dot{q}_2 + \mathrm{L}_1\, c_3\, c_3\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_3) \\ -s_5(-\mathrm{L}_2\, s_4\, \dot{q}_2 - \mathrm{L}_2\, s_4\, \dot{q}_3) + c_5\,(\mathrm{L}_1\, s_3\, s_3\, \dot{q}_2 + \mathrm{L}_1\, c_3\, c_3\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q_3}) \\ -\mathrm{L}_1\, c_3\, \dot{q}_1 - \mathrm{L}_2\, c_{34}\, \dot{q}_1 \end{bmatrix}$$

$$+ \begin{bmatrix} s_{345}\dot{q}_1 \\ c_{345}\,\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 + \dot{q}_4 + \dot{q}_5 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_4 \end{bmatrix} =$$

$$= \begin{bmatrix} -\mathrm{L}_2\, c_5\, s_4\, \dot{q}_2 - \mathrm{L}_2\, s_4\, c_5\, \dot{q_3} - \mathrm{L}_3 c_5\, s_5(\dot{q}_2 + \dot{q}_3 + \dot{q}_4) + s_5(\mathrm{L}_1\, s_3\, s_3\, \dot{q}_2 + \mathrm{L}_1\, c_3\, c_3\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_3) + L_4\, c_{345}\, \dot{q}_1 \\ -s_5(-\mathrm{L}_2\, s_4\, \dot{q}_2 - \mathrm{L}_2\, s_4\, \dot{q}_3) + c_5\,(\mathrm{L}_1\, s_3\, s_3\, \dot{q}_2 + \mathrm{L}_1\, c_3\, c_3\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q}_2 + \mathrm{L}_2\, c_4\, \dot{q_3}) - L_4\, s_{345}\, \dot{q}_1 \\ -\mathrm{L}_1\, c_3\, \dot{q}_1 - \mathrm{L}_2\, c_{34}\, \dot{q}_1 \end{bmatrix} \quad (16)$$

VI. Geometric Jacobian

The direct kinematic function for the manipulator is represented by the homogeneous transformation matrix $T_e^b(q)$, which specifies the position and orientation of the end effector relative to the reference base.

$$T_e^b(q) = \begin{bmatrix} n_e^b(q) & o_e^b(q) & a_e^b(q) & p_e^b(q) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (17)$$

where $n_e, o_e, and\ a_e$ are the unit vectors of a frame related to the end effector, and p_e is the position vector of the origin of such a frame with respect to the base frame. q is the $(nx1)$ vector of joint variables.

Determining the link between the joint velocities and the end-effector linear and angular velocities is the aim of differential kinematics. The mapping is defined by a matrix known as the geometric Jacobian, which is dependent on the configuration of the manipulator. In general, we describe end effector linear velocity $\dot{p}$ and angular velocity ω as functions of joint velocities $\dot{q}$.

$$V_{(6\times1)} = \begin{bmatrix} \dot{q} \\ \omega \end{bmatrix}_{(6\times1)} = J(q)\,\dot{q} = \begin{bmatrix} J_{P(3\times n)} \\ J_{O(3\times n)} \end{bmatrix}_{(6\times n)} \dot{q}_{(n\times1)} \quad (18)$$

The $(6\times n)$ matrix J is the manipulators geometric Jacobian, which is a function of the joint variables in general.

1 Jacobian Computation

This section presents the derivation of the Jacobian matrix for six degrees of freedom with a prismatic joint. The manipulator is comprised of five revolute joints and one prismatic joint (n = 6). In this context, we will outline the systematic process of deriving Jacobian matrices. We have shown that the Jacobian matrices can be represented as $(6\times n)$ matrices. The vector J can be divided into three column vectors. The vector J can be divided into (3×1) three column vectors. J matrices in our manipulator have the dimensions (6×6). Additionally, it can be displayed as:

$$J = \begin{bmatrix} J_{p1} & J_{p2} & J_{p3} & J_{p4} & J_{p5} & J_{p6} \\ J_{O1} & J_{O2} & J_{O3} & J_{O4} & J_{O5} & J_{O6} \end{bmatrix} \qquad (19)$$

$$J(q) = \begin{bmatrix} z_0 \times (p_e - p_0) & z_1 \times (p_e - p_1) & z_2 \times (p_e - p_2) & z_3 \times (p_e - p_3) & z_4 \times (p_e - p_4) & z_5 \\ z_0 & z_1 & z_2 & z_3 & z_4 & 0 \end{bmatrix}$$

The Jacobian can be calculated from the following equation:

$$J_{pi} = \begin{cases} z_{i-1} \times (p - p_{i-1}); & \text{for revolute joint i} \\ z_{i-1}; & \text{for prismatic joint i} \end{cases} \qquad (20)$$

$$J_{Oi} = \begin{cases} z_{i-1}; & \text{for revolute joint i} \\ 0; & \text{for prismatic joint i} \end{cases} \qquad (21)$$

Where P represents the end effector's position in relation to the base reference, as shown in Figure 5. p_{i-1} is the position of each revolute joint in relation to the base frame, which can be calculated using the first three elements of the fourth column of the transformation matrix T_n^0.

$$p_{i-1} = R_1^0(q_1)\ldots R_{i-1}^{i-2}(q_{i-1})\, p_0 \qquad (22)$$

Where $i = 1{:}6$ and $p_0 = [0 \quad 0 \quad 0 \; 1]^T$ allows selecting the fourth desired column.

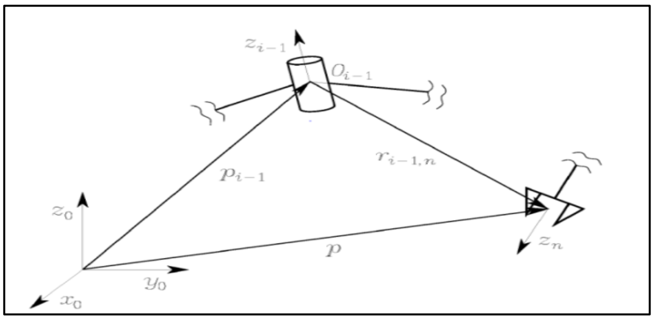

Fig. 5. Vectors needed for Jacobian computation

The z_{i-1} is the joint axis vector of each revolute joint and can be expressed by:

$$z_{i-1} = R_1^0(q_1)\ldots R_{i-1}^{i-2}(q_{i-1})\, z_0 \qquad (23)$$

Where $i = 1{:}6$ and $z_0 = [0 \quad 0 \quad 1]^T$ allows selecting the third desired column.

$$z_0 = [0 \quad 0 \quad 1]^T$$

$$z_1 = [0 \quad 0 \quad 1]^T$$

$$z_2 = [s_{12} \quad -c_{12} \quad 0]^T$$

$$z_3 = [s_{12} \quad -c_{12} \quad 0]^T$$

$$z_4 = [s_{12} \quad -c_{12} \quad 0]^T$$

$$z_5 = [s_{12} \quad -c_{12} \quad 0]^T$$

When the joints are revolute the $p - p_{i-1}$ must be calculated from the Eq (24)

$$p - p_{i-1} = A_0^1 \ldots A_n^{n-1}\bar{x} - A_0^1 \ldots A_{i-1}^{i-2}\,\bar{x} \qquad (24)$$

$\bar{x}$ is equal to the vector $[0 \quad 0 \quad 0 \quad 1]^T$. So,

$$(p_e - p_0)_{Revolute} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$(p_e - p_1)_{Revolute} = \begin{bmatrix} p_x \\ p_y \\ p_z - H \end{bmatrix}$$

$$(p_e - p_2)_{Revolute} = \begin{bmatrix} p_x \\ p_y \\ p_z - H \end{bmatrix}$$

$$(p_e - p_3)_{Revolute} = \begin{bmatrix} p_x - L_1 c_3 c_{12} \\ p_y - L_1 c_3 s_{12} \\ p_z - L_1 s_3 - H \end{bmatrix}$$

$$(p_e - p_4)_{Revolute} = \begin{bmatrix} p_x - c_{12}\,(L_2 c_{34} + L_1 c_3) \\ p_y - s_{12}\,(L_2 c_{34} + L_1 c_3) \\ p_z - H - L_2 s_{34} - L_1 s_3 \end{bmatrix}$$

$$For\ prismatic\ join\ (p_e - p_5)_{prismatic}\ not$$

2 SINGULARITY ANALYSIS

- **At joint 1** (Revolute joint): In the provided mathematical notation related to singularity analysis for a robotic manipulator with a revolute joint (joint 1), joint 1's position and Jacobian matrices for linear and angular velocities are defined.

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (p_e - p_0) = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \ z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{L_1} = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (p_e - p_0)\right] = \begin{bmatrix} -p_y \\ p_y \\ 0 \end{bmatrix}, \quad J_{A_1} = z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} J_{L_1(Linear)} \\ J_{A_1(angular)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (p_e - p_0) \\ z_0 \end{bmatrix} = \begin{bmatrix} -p_y \\ p_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (25)$$

- **At joint 2** (Revolute joint):

$$A_1 = T_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & H \\ 0 & 0 & 0 & 1 \end{bmatrix}, p_1 = \begin{bmatrix} 0 \\ 0 \\ H \end{bmatrix}, (p_e - p_1) = \begin{bmatrix} p_x \\ p_y \\ p_z - H \end{bmatrix}, \ z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{L_2} = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (p_e - p_1)\right] = \begin{bmatrix} -p_x \\ p_y \\ 0 \end{bmatrix}, \quad J_{A_2} = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} J_{L2} \\ J_{A2} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (p_e - p_1) \\ z_1 \end{bmatrix} = \begin{bmatrix} -p_y \\ p_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (26)$$

- **At joint 3** (Revolute joint):

$$A_2 = T_1^0 . T_2^1 = \begin{bmatrix} - & - & s_{12} & 0 \\ - & - & -c_{12} & 0 \\ - & - & 0 & \mathrm{H} \\ - & - & - & 1 \end{bmatrix}, p_2 = \begin{bmatrix} 0 \\ 0 \\ H \end{bmatrix}, \ (p_e - p_2) = \begin{bmatrix} p_x \\ p_y \\ p_z - H \end{bmatrix}, z_2 = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix}$$

$$J_{L_3} = \left[\begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix} \times (p_e - p_2) \right] = \begin{bmatrix} -c_2\,(p_z - H) \\ -s_2\,(p_z - H) \\ s_2\, p_y - c_2\, p_x \end{bmatrix}, \quad J_{A_3} = \mathrm{z}_2 = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} J_{L_3} \\ J_{A_3} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix} \times (p_e - p_2) \\ \mathrm{z}_2 \end{bmatrix} = \begin{bmatrix} -c_{12}\,(p_z - H) \\ -s_{12}\,(p_z - H) \\ s_{12}\, p_y + c_{12}\, p_x \\ s_{12} \\ -c_{12} \\ 0 \end{bmatrix} \qquad (27)$$

- **At joint 4** (Revolute joint):

$$A_3 = T_1^0 . T_2^1 . T_3^2 = \begin{bmatrix} - & - & s_{12} & L_1 c_3 c_{12} \\ - & - & -c_{12} & L_1 c_3 s_{12} \\ - & - & 0 & H + L_1 s_3 \\ - & - & - & 1 \end{bmatrix}, p_3 = \begin{bmatrix} L_1 c_3 c_{12} \\ L_1 c_3 s_{12} \\ H + L_1 s_3 \end{bmatrix}, (p_e - p_3) = \begin{bmatrix} p_x - L_1 c_3 c_{12} \\ p_y - L_1 c_3 s_{12} \\ p_z - L_1 s_3 - H \end{bmatrix}$$

$$J_{L_4} = \begin{bmatrix} -c_{12}\,(p_z - L_1 s_3 - H) \\ -s_{12}(p_z - L_1 s_3 - H) \\ s_{12}(p_y - L_1 c_3 s_{12}) + c_{12}(p_x - L_1 c_3 c_{12}) \end{bmatrix}, \quad J_{A_4} = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix}, \ z_3 = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} J_{L_4} \\ J_{A_4} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix} \times (p_e - p_3) \\ \mathrm{z}_3 \end{bmatrix} = \begin{bmatrix} -c_{12}\,(p_z - L_1 s_3 - H) \\ -s_{12}(p_z - L_1 s_3 - H) \\ s_{12}(p_y - L_1 c_3 s_{12}) + c_{12}(p_x - L_1 c_3 c_{12}) \\ s_{12} \\ -c_{12} \\ 0 \end{bmatrix} \qquad (28)$$

- **At joint 5** (Revolute joint):

$$A_4 = T_1^0 . T_2^1 . T_3^2 . T_4^3 = \begin{bmatrix} - & - & s_{12} & c_{12}\,(L_2 c_{34} + L_1 c_3) \\ - & - & -c_{12} & s_{12}\,(L_2 c_{34} + L_1 c_3) \\ - & - & 0 & H + L_2 s_{34} + L_1 s_3 \\ - & - & - & 1 \end{bmatrix}, p_4 = \begin{bmatrix} c_{12}\,(L_2 c_{34} + L_1 c_3) \\ s_{12}\,(L_2 c_{34} + L_1 c_3) \\ H + L_2 s_{34} + L_1 s_3 \end{bmatrix},$$

$$(p_e - p_4) = \begin{bmatrix} p_x - c_{12}\,(L_2 c_{34} + L_1 c_3) \\ p_y - s_{12}\,(L_2 c_{34} + L_1 c_3) \\ p_z - H - L_2 s_{34} - L_1 s_3 \end{bmatrix}, \qquad z_4 = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix}$$

$$J_{L_5} = \begin{bmatrix} -c_{12}(p_z - H - L_2 s_{34} - L_1 s_3) \\ -s_{12}(p_z - H - L_2 s_{34} - L_1 s_3) \\ s_{12}\,(p_y - s_{12}\,(L_2 c_{34} + L_1 c_3)) + c_{12}(p_x - c_{12}\,(L_2 c_{34} + L_1 c_3)) \end{bmatrix}, J_{A_5} = z_4 = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix}$$

$$J_5 = \begin{bmatrix} J_{L_5} \\ J_{A_3} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix} \times (p_e - p_4) \\ \mathrm{z}_4 \end{bmatrix} = \begin{bmatrix} -c_{12}(p_z - H - L_2 s_{34} - L_1 s_3) \\ -s_{12}(p_z - H - L_2 s_{34} - L_1 s_3) \\ s_{12}\,(p_y - s_{12}\,(L_2 c_{34} + L_1 c_3)) + c_{12}(p_x - c_{12}\,(L_2 c_{34} + L_1 c_3)) \\ s_{12} \\ -c_{12} \\ 0 \end{bmatrix} (29)$$

- **At joint 6** (Revolute joint):

$$\mathrm{A}_5 = T_1^0.\, T_2^1.\, T_3^2.\, T_4^3.\, T_5^4 = \begin{bmatrix} - & - & s_{12} & - \\ - & - & -c_{12} & - \\ - & - & 0 & - \\ - & - & - & 1 \end{bmatrix}, z_{i-1\,[\text{Prismatic joint}]} = z_5 = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix}$$

$$J_{L_6} = z_5 = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \end{bmatrix}, \quad J_{A_6} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_6 = \begin{bmatrix} J_{L_6} \\ J_{A_6} \end{bmatrix} = \begin{bmatrix} z_5 \\ 0 \end{bmatrix} = \begin{bmatrix} s_{12} \\ -c_{12} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (30)$$

Now we can write the Jacobian matrix as shown in equation (30):

$$\mathrm{J} = [\, \mathrm{J}_1 \;\; \mathrm{J}_2 \;\; \mathrm{J}_3 \;\; \mathrm{J}_4 \;\; \mathrm{J}_5 \;\; \mathrm{J}_6 \,] \qquad (31)$$

The linear Jacobian J_L and angular Jacobean J_{A}, will become:

$$\mathrm{J} = \begin{bmatrix} J_{Linear} \\ J_{Angular} \end{bmatrix} = \begin{bmatrix} \mathrm{J_{L1}} & \mathrm{J_{L2}} & \mathrm{J_{L3}} & \mathrm{J_{L4}} & \mathrm{J_{L5}} & \mathrm{J_{L6}} \\ \mathrm{J_{A1}} & \mathrm{J_{A2}} & \mathrm{J_{A3}} & \mathrm{J_{A4}} & \mathrm{J_{A5}} & \mathrm{J_{A6}} \end{bmatrix} \qquad (32)$$

$$J_{\mathrm{L}} = \begin{bmatrix} -p_y & -p_y & -c_{12}\,(p_z - H) & -c_{12}\,(p_z - L_1 s_3 - H) & -c_{12}(p_z - H - L_2 s_{34} - L_1 s_3) & s_{12} \\ p_x & p_x & -s_{12}\,(p_z - H] & -s_{12}(p_z - L_1 s_3 - H) & -s_{12}(p_z - H - L_2 s_{34} - L_1 s_3) & -c_{12} \\ 0 & 0 & s_{12}\,p_y + c_{12}\,p_x & s_{12}(p_y - L_1 c_3 s_{12}) + c_{12}(p_x - L_1 c_3 c_{12}) & s_{12}\,(p_y - s_{12}\,(L_2 c_{34} + L_1 c_3)) + c_{12}(p_x - c_{12}\,(L_2 c_{34} + L_1 c_3)) & 0 \end{bmatrix}$$

$$J_{\mathrm{A}} = \begin{bmatrix} 0 & 0 & s_{12} & s_{12} & s_{12} & 0 \\ 0 & 0 & -c_{12} & -c_{12} & -c_{12} & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (33)$$

The general Jacobian become J:

$$= \begin{bmatrix} -p_y & -p_y & -c_{12}\,(p_z - H) & -c_{12}\,(p_z - L_1 s_3 - H) & -c_{12}(p_z - H - L_2 s_{34} - L_1 s_3) & s_{12} \\ p_x & p_x & -s_{12}\,(p_z - H) & -s_{12}(p_z - L_1 s_3 - H) & -s_{12}(p_z - H - L_2 s_{34} - L_1 s_3) & -c_{12} \\ 0 & 0 & s_{12}\,p_y + c_{12}\,p_x & s_{12}(p_y - L_1 c_3 s_{12}) + c_{12}(p_x - L_1 c_3 c_{12}) & s_{12}\,(p_y - s_{12}(L_2 c_{34} + L_1 c_3)) + c_{12}(p_x - c_{12}(L_2 c_{34} + L_1 c_3)) & 0 \\ 0 & 0 & s_{12} & s_{12} & s_{12} & 0 \\ 0 & 0 & -c_{12} & -c_{12} & -c_{12} & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general Jacobian become J:

$$\mathrm{J} = \begin{bmatrix} \mathrm{J}_{11} & \mathrm{J}_{12} & \mathrm{J}_{13} & \mathrm{J}_{14} & \mathrm{J}_{15} & \mathrm{s}_{12} \\ \mathrm{J}_{21} & \mathrm{J}_{22} & \mathrm{J}_{23} & \mathrm{J}_{24} & \mathrm{J}_{25} & -\mathrm{c}_{12} \\ 0 & 0 & \mathrm{J}_{33} & \mathrm{J}_{34} & \mathrm{J}_{35} & 0 \\ 0 & 0 & s_{12} & s_{12} & s_{12} & 0 \\ 0 & 0 & -c_{12} & -c_{12} & -c_{12} & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{J}_{11} = -(\mathrm{L}_3\, \mathrm{s}_{12}\;\, \mathrm{c}_{345} - \mathrm{L}_4\, \mathrm{c}_{12} + \mathrm{s}_{12}(\, \mathrm{L}_2\, \mathrm{c}_{34} + \mathrm{L}_1\, \mathrm{s}_3))$$

$$\mathrm{J}_{12} = -(\mathrm{L}_3\, \mathrm{s}_{12}\;\, \mathrm{c}_{345} - \mathrm{L}_4\, \mathrm{c}_{12} + \mathrm{s}_{12}(\, \mathrm{L}_2\, \mathrm{c}_{34} + \mathrm{L}_1\, \mathrm{s}_3))$$

$$\mathrm{J}_{13} = -c_{12}\,(\mathrm{L}_2\, \mathrm{s}_{34} + \mathrm{L}_1\, \mathrm{s}_3 + \mathrm{L}_4 - H)$$

$$\mathrm{J}_{14} = -c_{12}(L_2\, s_{34} + \mathrm{L}_4 - \mathrm{H})$$

$$\mathrm{J}_{15} = -\,\mathrm{c}_{12}(\mathrm{L}_4 - \mathrm{H})$$

$$\mathrm{J}_{21} = \mathrm{L}_3\, \mathrm{c}_{12}\;\, \mathrm{c}_{345} + \mathrm{L}_4\, \mathrm{s}_{12} + \mathrm{c}_{12}(\, \mathrm{L}_2\, \mathrm{s}_{34} + \mathrm{L}_1\, \mathrm{c}_3)$$

$$\mathrm{J}_{22} = \mathrm{L}_3\, \mathrm{c}_{12}\;\, \mathrm{c}_{345} + \mathrm{L}_4\, \mathrm{s}_{12} + \mathrm{c}_{12}(\, \mathrm{L}_2\, \mathrm{s}_{34} + \mathrm{L}_1\, \mathrm{c}_3)$$

$$J_{23} = -s_{12}\,(L_2\,s_{34} + L_1\,s_3 + L_4 - H)$$

$$J_{24} = -\,s_{12}\,(L_2\,s_{34} + L_4 - H)$$

$$J_{25} = -\,s_{12}(L_4 - H)$$

$$J_{26} = -c_{12}$$

$$J_{33} = s_{12}\,(L_3\,s_{12}\,\ c_{345} - L_4\,c_{12} + \ s_{12}(\,L_2\,c_{34} + L_1\,s_3)) + c_{12}\,(L_3\,c_{12}\,\ c_{345} + L_4\,s_{12} + \ c_{12}(\,L_2\,s_{34} + L_1\,c_3))$$

$$J_{34} = s_{12}(L_3\,s_{12}\,\ c_{345} - L_4\,c_{12} + \ s_{12}(\,L_2\,c_{34} + L_1\,s_3) - L_1c_3s_{12}) + \ c_{12}(L_3\,c_{12}\,\ c_{345} + L_4\,s_{12} + \ c_{12}(\,L_2\,s_{34} + L_1\,c_3) - L_1c_3c_{12})$$

$$J_{35} = s_{12}\,(L_3\,s_{12}\,\ c_{345} - L_4\,c_{12} + \ s_{12}(\,L_2\,c_{34} + L_1\,s_3) - s_{12}\,(L_2c_{34}\, + L_1c_3)) + c_{12}\,(L_3\,c_{12}\,\ c_{345} + L_4\,s_{12} + \ c_{12}(\,L_2\,s_{34} + L_1\,c_3) - c_{12}\,(L_2c_{34}\, + L_1c_3))$$

VII. Kinematic Singularity

Singularities in manipulators have non-local implications and arise from derivatives rank deficiency. There are different types of singularities for serial and parallel manipulators, and their analysis is important for engineering. For serial manipulators, it is the singularities of the kinematic mapping/forward kinematics and trajectories that are of interest, whereas for fully parallel manipulators it is those of the constraint function defining the configuration space and of the projection onto the articular space (inverse kinematics). The meaning of singularities in engineering has several aspects:

1) Loss of freedom: The derivative of kinematic mapping, also known as forward kinematics, is the process of converting joint velocities into generalised end-effector velocities, which include both linear and angular velocities. In the robotics literature, this linear transformation is commonly referred to as the manipulator Jacobian. A decrease in rank results in a reduction of the image's dimension, signifying a reduction of one or more degrees in the end effector's instantaneous motion.

2) Workspace: When a manipulator is at a boundary point of its workspace, the manipulator is necessarily at a singular point of its kinematic mapping, though the converse is not the case. Interior components of the singular set separate regions with different numbers or topological types of inverse kinematics. These are usually associated with a change of posture in some component of the manipulator. Therefore, knowledge of the manipulator singularities provides valuable information about its workspace [6].

3) Loss of control: In close proximity to a singularity, this matrix is ill-defined. If the control algorithm fails, the joint velocities and accelerations may reach levels that are not sustainable. On the other hand, force control techniques that are suitable for parallel manipulators may lead to excessive joint forces or torques when the projection into the joint space approaches singularities.

VIII. MATLAB CODE EXPLANATION

1 Transfer matrix's function

A function in MATLAB is defined to accept the D-H parameters as input and give the value A_{i-1}^{i-2} as output:

```
%%%%%% Trans.m %%%%%%
function [A]= DH(a, alpha, d, theta)
% D-H Homogeneous Trasnformation Matrix (a alpha d theta)
A = [cos(theta) -sin(theta)*round(cos(alpha)) sin(theta)*round(sin(alpha))
    a*cos(theta); sin(theta) cos(theta)*round(cos(alpha))
    -cos(theta)*round(sin(alpha)) a*sin(theta); 0 round(sin(alpha))
    round(cos(alpha)) d; 0   0   0  1];
end
```

2 Inserting D-H Parameters

In this part, we manually input the Denavit-Hartenberg (D-H) parameters into the code.


```
% Inverting D-H convention parameters
A1=Trans(0, 0, H, th1);
A2=Trans(0, pi/2, 0, th2);
A3=Trans(L1, 0, 0, th3);
A4=Trans(L2, 0, 0, th4);
A5=Trans(L3, 0, 0, th5);
A6=Trans(0, pi/2, L4, th6);
```

3 Creating Transfer Matrices

By determining a transfer matrix for each transformation and post multiplying them, we obtain the transformation matrix for A_{ee}^{0}, which is referred to as T_6.

```
% Creating Transfer matrices
T2=A1*A2;
T3=A1*A2*A3;
T4=A1*A2*A3*A4;
T5=A1*A2*A3*A4*A5;
T6=A1*A2*A3*A4*A5*A6;
```

4 Creating p_{i-1}, z_{i-1}, and P

To compute the Jacobian matrices, we generate the p_{i-1}, z_{i-1}, and P matrices.

```
% Creating zi

z0= [0;0;1];
z1= A1(1:3,3);
z2= T2(1:3,3);
z3= T3(1:3,3);
z4= T4(1:3,3);
z5= T5(1:3,3);

% Creating pi

p0=[0;0;0];
p1=A1(1:3,4);
p2=T2(1:3,4);
p3=T3(1:3,4);
p4=T4(1:3,4);
p5=T5(1:3,4);

P=T6(1:3,4);
```

5 Jacobian Matrix computation

In this part of program, we compute the Jacobian matrix . To get equations that are more simple terms, we apply the simplify command. To obtain the decoupled singularities, the Jacobians of the (3×3) blocks are calculated.

```
% Jacobian matrix Computation
J= simplify(
[cross(z0,P-p0),cross(z1,P-p1),
cross(z2,P-p2),cross(z3,P-p3),
cross(z4,P-p4),cross(z5,P-p5);
z0      ,      z1      ,       z2
z3      ,      z4      ,      z5
])

% (3*3) blocks Jacobians
J11=J(1:3,1:3);
J22=J(4:6,4:6);
```

6 Jacobian Matrice Determinant

In the end, the determinant for each Jacobian is computed. Streamlining commands aid in achieving a more concise outcome.

```
% Determinant Calculation
det00=simplify(det(J));
det11=simplify(det(J11));
det22=simplify(det(J22));
```


The determinant of the Jacobian matrix for your RRRRRP manipulator is:

Det J = a2 a3 sin(theta3) sin (theta 2+theta3 + theta4) cos(theta5)

Singularities occur when:

- sin(theta3) =0, theta3 = 0, π
- sin (theta 2+theta3 + theta4) =0, theta 2+theta3 + theta4=0, π
- cos(theta5) =0, theta5= -+π/2

Below is a table of results for singularity conditions in the RRRRRP manipulator.

Singularity	Description	Condition
Wrist Singularity (Axes Alignment)	Occurs when the axes of the last three joints (4, 5, 6) align.	Θ5 = - π/2 or + π/2
Elbow Singularity Fully Extended	Fully Extended	θ2+θ3+θ4=0 Fully Extended
Elbow Singularity Fully Folded	Fully Folded	θ2+θ3+θ4= π Fully Folded
Base Singularity	Base joint aligned with arm direction	θ1 = 0 or θ1 = π
Planar Singularity	The manipulator's links become collinear, reducing degrees of freedom.	θ2+θ3+θ4= - π, + π
Workspace Singularity	Prismatic Joint Fully Extended	$d_5 = d_{min}$, $d_5 = d_{max}$ Fully retracted, fully extended

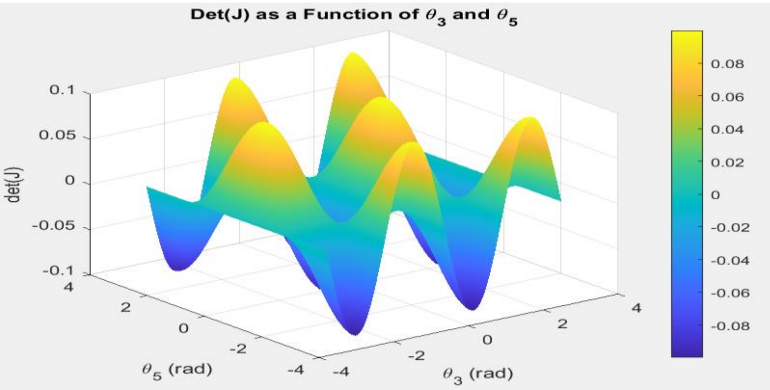

Fig. 3. 3D Surface Plot (theta_3) vs. (theta_5)

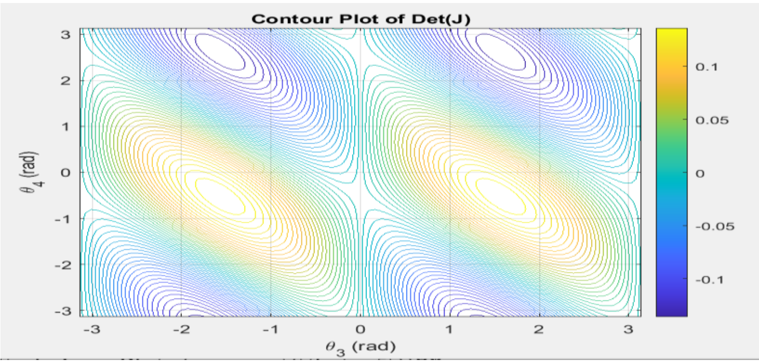

Fig. 4. Contour Plot theta_3 vs. theta_4

CONCLUSION:

In this article, we have thoroughly examined the geometric Jacobians and kinematic singularities of a 6-DOF robotic manipulator, including a prismatic joint. Through the systematic derivation of the Jacobian matrix, we have demonstrated the relationship between joint velocities and the end-effector's linear and angular velocities. The analysis highlighted the occurrence of rank deficiencies in the Jacobian, which correspond to critical kinematic singularities. These singularities, identified through the determinant of the Jacobian, restrict the manipulator's movement and can lead to potential control issues.

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