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NOVEL SEQUENCES OF PALINDROMIC PRIMES IN VARIOUS BASES

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Abstract: Palindromic numbers have intrigued amateur mathematicians and number theorists alike. Initially regarded as recreational mathematics, these numbers have been extensively explored and are well-documented in the On-Line Encyclopedia of Integer Sequences (OEIS). Within the last few decades, an exhaustive list of sequences has been compiled regarding palindromic numbers, notably palindromic primes and appended palindromic numbers. Both palindromic primes and appended palindromic numbers have been well-studied within bases 2-10, yet are poorly documented in larger bases. To extend the literature on appended palindromic numbers, a novel algorithm is proposed that computes sequences of primes with prime mirrors in bases 2-62, resulting in 52 novel sequences. A second algorithm is proposed that computes the list of primes that require an additional base to obtain a prime mirror, providing yet another novel sequence.

Keywords: prime, palindrome, base, conversion, cryptography

I. INTRODUCTION

Palindromes have captivated linguists for centuries with analyses dating back to Henry Peacham's thesis in 1643 [1]. His definition, still in use, states that a palindrome is a string of letters that reads the same forwards or backwards (i.e., level, stats, radar). However, palindromes are not limited to words. Mathematicians have compiled a list of palindromic numbers, such as 0, 6, 88, 101, 87678, and more.

It did not take long for number theorists to generate subsets of palindromic numbers, which are well-documented in the On-Line Encyclopedia of Integer Sequences (OEIS). One of the more popular subsets of the sequence of palindromic numbers in base 10 is palindromic squares: 0, 1, 4, 9, 121, … (see OEIS sequence A002779). A sequence that is of more interest to this paper is the sequence of palindromic primes in base 10: 2, 3, 5, 7, 11, 101, 131, 151, 181, ... (sequence A002113).

Since the publication of sequence A002113 in 2000, mathematicians and computer scientists have reported sequences of palindromic primes in bases 2-16, recorded in base 10 (sequences A016041, A029971-A029982, A007500, A029732). For example, converting 17 into base 2 becomes 10001. If we convert 911 into base 5, we get 12121. Once we convert 1621 into base 13, we obtain the palindromic 979. To take this one step further, researchers have reported those same sequences written in their respective bases rather than being written in base 10.

Recently, there has been interest in appended palindromic numbers (APNs). These APNs have an even number of digits such that the first half of digits are a prime number in base 10 and the second half of digits are a prime number in base 10. Consider 1771 in base 30. The first half of digits, 17, is equivalent to 37 in base 10, whereas the second half of digits, 71, is equivalent to 211 in base 10. Since 37 and 211 are both prime, 1771 is an APN in base 30.

To contribute to the blossoming list of palindrome-related sequences, this paper proposes a novel algorithm which has discovered sequences of primes with prime mirrors throughout bases 11-62. However, the main contribution of this paper is the *Smaug* sequence, which contains the prime numbers that require the use of an additional base to have a prime mirror.

In Section II, a brief literature review is provided that covers introductory reviews in palindromic primes, interesting applications and subsets, and sequences that are related to the novel palindromic sequences that this paper presents. In Section III, a formal definition of palindromic numbers and prime palindromic numbers are provided, as well as a brief discussion of the use of alphabetical characters for integers in bases greater than 10. In Section IV, an algorithm is proposed that generates novel sequences of primes with prime mirrors in various bases. An additional algorithm is proposed that generates the *Smaug* sequence of prime numbers. The generated sequences are presented in Table I. In Section V, the paper is concluded with a few remarks that discuss ideas for more novel sequences related to palindromic primes.

II. LITERATURE REVIEW

The beginning of the palindromic prime journey does not have a definitive date. However, Gabai and Coogan published a brief, yet thorough guide to palindromic primes in 1969 [2]. In the following decades, many papers on palindromic primes have been published. Some discuss various properties of this charming sequence, such as their formation of a palindromic prime pyramid [3], their connection to experimental number theory [4], and a subsequence called palindromic Smith numbers [5].

If Gabai and Coogan's review was not sufficient, one could review some of the other introductions to palindromic primes [6-7]. However, for the true palindrome aficionados, the comprehensive encyclopedia of prime numbers by Ribenboim may be an excellent read [8].

As a popular topic in recreational mathematics, palindromic primes have intrigued many for decades. Among those that explored the beauty of palindromic numbers was famous architect Buckminster Fuller. He identified that palindromic primes make up a subset of his Scheherazade numbers, affably known as the Scheherazade Sublimely Rememberable Comprehensive Dividends [9]. More recently, Cilleruelo and his coauthors discovered that every positive integer can be written as a sum of three palindromic numbers [10].

Perhaps one of the more interesting applications of palindromic primes is found in the Lychrel numbers, a subset of palindromic integers [11]. A Lychrel number is one that never becomes palindromic through the Lychrel process, which consists of reversing the number and appending a known palindromic number.

Several sequences have been recorded relating to APNs. The OEIS sequence A074832 contains primes whose binary mirror is also prime: 3, 5, 7, 11, 13, 17, 23, 29, 31, and so on. For example, thirteen in base 2 is 1101 with mirror 1011. Once 1011 is converted back to base 10, 11 is obtained. Primes with prime mirrors in base 3-10 have also been recorded (sequences A074833-A074834, A075235-A075239). However, a similar sequence for other bases has not been reported.

III. DEFINITIONS

A. *Palindromic Numbers and Primes*

Consider a number $n > 0$ with digits denoted a_i (i=0,…,k) in base $b \geq 2$. Using standard notation, we can write n as

$$n=a_0b^0 + a_1b^1+\ldots+ a_kb^k \quad (1)$$

where $0 \leq a_i < b$ for all i and $a_k \neq 0$. Then, we say that n is a palindromic number if and only if $a_i = a_{k-i}$. Additionally, a palindromic prime is simply a palindromic number that is only divisible by 1 and itself.

B. *Alphabetic Characters in Bases*

When working in bases greater than 10, a distinction must be made between a two-digit number and a character with that same value. A familiar example might be values in the hexadecimal system. Consider converting 11 into base 16. Since 11 < 16, the value of 11 should be written as one digit; however, our perception of the value of 11 in base 10 necessitates two digits. Since we cannot write the value of 11 as another number, mathematicians elected to use alphabetical characters. So, in base 16 (and every other base greater than 10), the value of 11 is recorded as 'B.' Likewise, the value of 10 is recorded as 'A' and 12 as 'C.'

In the hexadecimal system, six uppercase letters are used to represent 10, 11, 12, 13, 14, and 15. Assuming that we only use 26 uppercase letters, 26 lowercase letters and 10 digits (i.e., 0, 1, 2, …, 9), we have a total of 62 possible symbols to represent different values. Therefore, most mathematicians and computer scientists limit base conversions between base 2 and base 62.

IV. ALGORITHMS AND SEQUENCES

A. *Algorithm I*

Until now, the list of prime sequences with prime mirrors was limited to bases 2-10. In Fig. 1, an algorithm is provided in pseudocode that obtains similar sequences in bases 11-62. While cycling through each integer in the desired range within each base, we check if the integer is prime. If so, then that number is converted from base 10 into the current base and its mirror is obtained. Then, the mirror is converted back into base 10. If the converted mirror is prime, then we found a pair of prime numbers that form an APN in the current base. The output is printed, and the next integer is processed.

In this pseudocode, integer vectors are used; however, the vectors may be replaced by dynamic arrays if desired. The sequences generated from Algorithm I in bases 11-62 are novel and have not yet been recorded in OEIS. Since there is a total of 52 novel sequences, only a selection of them is reported in Table I.

Algorithm 1 Compute sequences of primes that have prime mirrors

```
obtain input of upper-bound of integers to process, called Max
N = {2, 3, ..., Max}
let B = {2, 3, ..., 62} be the set of possible bases
declare int vectors number and backwards
for each base b in B do
    for each number n in N do
        if n is prime then
            clear number and backwards
            convert n to base b and store in number
            store the mirror of number in backwards
            convert backwards into base 10 as mirror10
            if mirror10 is prime then
                print n, b, mirror10, number, and backwards
            end if
        end if
    end for
end for
```

Figure 1. The algorithm to obtain sequences of primes that have prime mirrors.

B. *Algorithm II*

In base 2, the sequence of primes with prime mirrors, written in base 10, is 3, 5, 7, 11, 13, 17, and so on. Note the absence of 2, which is included in the sequence of primes with prime mirrors in base 3. Additionally, note the absence of 59 in the sequence for bases 2-3 and its presence in the sequence for base 4. Every prime less than 59 has a prime mirror in bases 2-3. Yet, the prime 59 requires base 4 to have a prime mirror.

One charming sequence of integers may be the primes that require an additional base to have a prime mirror. Clearly, the first two values in the sequence would be 2, then 59. Due to the seemingly greedy nature of these primes, an apt name for this sequence is the *Smaug* sequence. Algorithm II obtains the Smaug sequence up to values less than the provided upper-bound.

In the example concerning 59, there was a prime mirror in neither base 2 nor base 3. We had to check both bases before considering base 4. Let B denote the highest base that we are currently considering. For each number less than the provided upper-bound, we set a Boolean flag to false, denoting that a prime mirror has yet to be found in any base. If the current number n is prime, then we determine if its mirror is prime in the current base b. If it is, then the Boolean flag is set to true. After iterating through each base less than or equal to B, we check if the flag is false. If so, a prime mirror was not found in any of the previously considered bases. So, B is incremented until a prime mirror is found, and n is printed to the screen.

The Smaug sequence has not been reported in any manuscript to date and makes its debut appearance at the bottom of Table I.

Base	***Sequence***
11	2 3 5 7 13 17 19 23 29 31 37 43 47 53 61 67 71 73 79 83 89 97 101 103 107 113
12	2 3 5 7 11 13 17 61 67 71 89 137 151 157 163 167 179 181 191 193 197 211 227
13	2 3 5 7 11 17 19 23 29 31 41 47 53 67 71 73 79 83 89 101 103 107 109 127 131
14	2 3 5 7 11 13 17 19 23 43 47 53 71 73 79 127 131 137 139 157 163 167 191 193
15	2 3 5 7 11 13 17 19 29 31 37 41 43 61 67 73 107 109 113 127 131 167 173 197
16	2 3 5 7 11 13 17 23 31 53 59 61 83 89 113 149 179 191 211 241 251 257 269
17	2 3 5 7 11 13 23 31 37 47 53 59 61 73 79 89 101 103 113 131 139 149 151 163
18	2 3 5 7 11 13 17 19 29 97 103 107 131 139 199 239 241 311 331 349 353 359
19	2 3 5 7 11 13 17 29 31 41 43 47 59 61 67 71 73 79 83 89 97 103 107 109 113
20	2 3 5 7 11 13 17 19 23 29 61 71 73 79 151 157 181 191 193 197 199 223 227
21	2 3 5 7 11 13 17 19 23 31 37 41 43 47 53 59 61 89 97 107 109 113 173 179 181
22	2 3 5 7 11 13 17 19 23 31 37 41 43 71 73 113 127 131 157 167 199 293 331 347
23	2 3 5 7 11 13 17 19 29 43 53 61 79 89 101 103 107 113 127 131 139 157 163
24	2 3 5 7 11 13 17 19 23 37 41 43 127 131 139 173 179 269 271 281 283 313 409
25	2 3 5 7 11 13 17 19 23 29 31 41 59 61 73 79 89 101 103 107 109 151 157 163
26	2 3 5 7 11 13 17 19 23 29 31 43 47 79 101 103 131 139 149 191 193 197 199
27	2 3 5 7 11 13 17 19 23 31 37 41 43 47 59 61 67 71 79 109 113 137 139 149 157
28	2 3 5 7 11 13 17 19 23 29 43 53 97 101 107 149 151 167 257 263 271 277 313
29	2 3 5 7 11 13 17 19 23 31 37 41 47 59 61 67 71 83 89 97 101 103 109 113 131
30	2 3 5 7 11 13 17 19 23 29 31 37 41 53 211 223 229 239 331 347 353 359 397
40	2 3 5 7 11 13 17 19 23 29 31 37 41 47 53 59 73 127 131 137 149 157 281 283
50	2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 71 73 79 89 97 151 157 163 167
60	2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 83 89 97 113 421 433
61	2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 67 73 89 101 127 139 157 163
62	2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 73 83 103 107 113 191
Smaug Sequence	2 59 109 137 593 769 1019 2137 8573 11801 47317 159059 169019 324449

Algorithm 2 Compute the Smaug sequence

```
obtain input of upper-bound of integers to process, called Max
N = {2, 3, ..., Max}
declare int vectors number and backwards
declare Boolean variable flag
initialize B = 2
for each number n in N do
    set flag = FALSE
    if n is prime then
        for each base b ≤ B do
            clear number and backwards
            convert n to base b and store in number
            store the mirror of number in backwards
            convert backwards into base 10 as mirror10
            if mirror10 is prime then
                set flag = TRUE
            end if
        end for
        if flag == FALSE then
            while flag == FALSE do
                increment B by one
                clear number and backwards
                convert n to base b and store in number
                store the mirror of number in backwards
                convert backwards into base 10 as mirror10
                if mirror10 is prime then
                    set flag = TRUE
                end if
            end while
            print n
        end if
    end if
end for
```

Figure 2. The algorithm to obtain the Smaug sequence.

V. CONCLUSION

With the use of these two algorithms, 53 novel sequences are obtained. However, additional novel sequences could be extracted from the first algorithm. Recall the discussion of alphabetical characters used in bases greater than 10. Suppose we want to obtain, in each base, a sequence of primes that have mirror primes where the APN does not require the use of an uppercase or lowercase alphabetical character. This can be achieved by checking if the vector *number* consists of solely Arabic numerals. If it does not, one can simply break out of the *if* statement on line 7 of Algorithm I.

The sequences reported in Table I are the primes that make up the first half of an APN when converted into their respective base. However, more sequences can be generated by reporting the resulting APNs in bases 11-62 rather than the primes, written in base 10.

Future exploration of prime palindromes can focus on their relation to other classes of primes. One possible connection that can be made is between prime palindromes and Catalan numbers. Additionally, a strong association between palindromic primes and twin primes is greatly needed. Perhaps the sequences provided in Table I may facilitate any connections to be made.

VI. Code Availability

Algorithms I and II are both available on GitHub (see github.com/michaelschwob/PrimePalindromes). Additionally, several related functions and files are provided. Most of the code used in this GitHub repository is C++ and the remainder is programmed in R.

VII. Acknowledgment

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