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THE (n-1)/2-REGULAR GRAPH ON n VERTICES

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Abstract. Let G be an undirected and simple graph on n vertices and degree of each vertex is equal $(n-1)/2$. We present some properties of G and confirm that G is a Hamiltonian graph.

Keywords. Regular graph, Hamiltonian graph, Petersen graph, Closure graph, Diameter of Graph

1. INTRODUCTION

Let $G=(V,E)$ be an undirected and simple graph on n vertices, where V be the vertex set and E be edge set of G. We use $|V|$ and $|E|$ to denote the number of vertices and the number edges of G, respectively. In G, the degree of vertex v is denoted by $\deg(v)$. The edge of two vertices u and v is denoted by (u,v) or uv. A graph is called *regular graph of degree* k (or k *- Regular graph*) if its vertices has degree k. We use $\delta(G)$ to denote the minimum degree of the vertices of G. The graph on n vertices with all vertices having degree $n-1$ is called the *complete graph* and denote by K_n.

A set of vertices in graph G is called *independent* if no two vertices in this set are non-adjacent. *Maximum independent set* is an independent set of largest possible size for a given graph. Denote by $\alpha(G)$ the size of a maximum independent set of G. A set $C \subseteq V$ is called *clique* if every two distinct vertices in C are adjacent in G

The graph $H=(W,F)$ is called a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$. Let v is a vertex of G, we use $G-v$ to denote the subgraph which obtained by deleting v from G. Livewise , if B is a set of vertices of G, graph $G-B$ is a subgraph of G whose obtained by deleting B from G.

We use $\omega(G)$ to denote the number of components of G. In G, a vertex v is called *cut vertex* if $\omega(G) < \omega(G-v)$. Denote by $G+uv$ the graph which obtained from G when previously non-adjacent vertices u and v are joined by a new edge uv. A set of vertices in a connected graph is called *disconnecting* if the graph becomes disconnected when this set is removed. Denote by $\kappa(G)$ the smallest size of a disconnecting set in G.

Graph G is called *1-tough* if $\omega(G-B) \leq |B|$ for every non-empty subset B of V.

The distance between two vertices in G is the number of edges in a shortest path connecting them. The *diameter* of G is the greatest distance between any pair of vertices and denote by $d(G)$.

A simple path in connected graph G that passes through every vertex exactly once is called *Hamiltonian path*. A simple cycle in a connected graph G that passes through every vertex exactly once is called *Hamiltonian cycle*. Any connected graph that contains a Hamiltonian cycle is called *Hamiltonian Graph*.

Recognizing Hamiltonian graph is hard problem. Now there are many theorems providing sufficient conditions for a graph to be Hamiltonian. Dirac [4] proved that if the minimum degree of the vertices of G is at least $n/2$ then G is Hamiltonian graph. Denote by $\sigma_2(G)$ - the degree sum of any two non-adjacent vertices in G. Ore [4] asserts results more generally, if $\sigma_2(G) \geq n$ then G is Hamiltonian graph. In [4], H. A. Jung proved that, if G is 1-tough and $\sigma_2(G) \geq n-4$, $n \geq 11$ then G is Hamiltonian graph.

In [1] and [2], we proved that, if $\sigma_2(G)=n-1$, there are three cases, if n is an even number then G is Non-Hamiltonian graph, if n is an odd number and

$2 < \alpha(G) < (n+1)/2$ then G is Hamiltonian graph, otherwise, G is Non-Hamiltonian graph.

In [5], Paul Erdos proved that, if ($n-2$)-Regular G graph with $|V(G)|=2n$ or $|V(G)|=2n-1$ and $\kappa(G)=2$, then, G is Hamiltonian if only if G is not the Petersen graph. Figure 1 is Petersen graph.

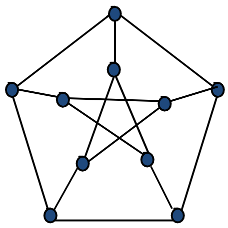

Figure 1. Petersen graph.

Bondy, Chvátal and Murty [3] used the definition on closure graph to define the necessary and sufficient condition for Hamiltonian graph. Following some sufficient conditions for Hamiltonian and non-Hamiltonian graph.

Theorem 1 (Bondy and Chvátal [3]). *Let G be a graph on n vertices and let u and v be nonadjacent vertices of G with degree sum at least n. Then, G is Hamiltonian graph if and only if $G+uv$ is Hamiltonian graph.*

Theorem 2 (Chvátal [3]). *If G is not 1-tough graph then G is not Hamiltonian graph.*

Denote by $Cl(G)$ the closure of G which derived from G by recursively joining pairs of nonadjacent vertices having degree sum at least n. Figure 2 illustrates graph G and its closure graph $Cl(G)$.

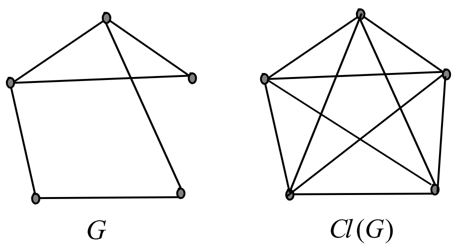

Figure 2

Theorem 3 (The Closure Lemma). *G is Hamiltonian if and only if $Cl(G)$ is Hamiltonian.*

Following result is special case of Theorem 3.

Corollary 1 (Bondy and Murty [3]). *If $Cl(G)$ is complete graph K_n then G is Hamiltonian.*

Theorem 4 (Nash-Williams, Bondy [5]). *If $\alpha(G) \le \delta(G)$, $\kappa(G) \ge 2$ and $\delta(G) \ge (n+2)/3$ then G is Hamiltonian.*

2. RESULT

Let G be an k-regular graph on n vertices, where $k=(n-1)/2$. Then, n must be an odd number and $\mathrm{mod}(n-1,4)=0$ (if not, $(n-1)/2=k$ be an odd number, i.e., graph G has number of vertices of odd degree is an odd number, this is absurd).

We use $G(n,k)$ to denote the set of k- regular graphs on n vertices, where $k=(n-1)/2$ and $\mathrm{mod}(n-1,4)=0$ (so, $n \ge 5$ and k be an even number). Figure 3 illustrates graphs in $G(5,2)$ and $G(9,4)$.

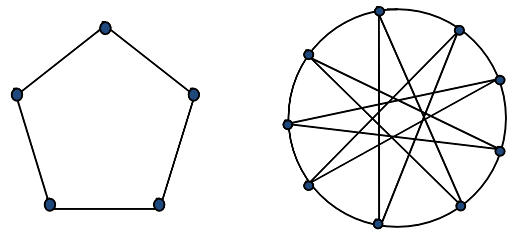

Figure 3. *Graphs in $G(5,2)$ and $G(9,4)$* **.**

Proposition 1. *For every $G \in G(n,k)$, G is connected graph.*

Proof. Suppose otherwise, G is disconnected graph. Let G^1 is a connected component of G and $|V(G^1)|=n_1$. Denote by G^2 the remaining of G and $|V(G^2)|=n_2$. We have $n_1+n_2=n$. Choose an any vertex u in G^1 and an any vertex v in G^2. Then, $(n-1)/2=\deg(u) \le n_1-1$, $(n-1)/2=\deg(v) \le n_2-1$. So, $n-1=\deg(u)+\deg(v) \le n_1-1+n_2-1=n-2$, a contradiction. Therefore, G is connected graph.

Proposition 2. *For every $G \in G(n,k)$, G contains a Hamiltonian path.*

Proof. Let u and v be any two non-adjacent vertices in G, we add an edge uv to G. Then, $\deg(u)=\deg(v)=1+(n-1)/2$. Let w is an any vertex such that w is non-adjacent to u or v of G, we have $\deg(w)+\deg(u) = (n-1)/2+1+(n-1)/2=n$ or

$\deg(w)+\deg(v)=(n-1)/2+1+(n-1)/2=n$. In other words, we add to the $G+uv$ graph the edges connecting two non-adjacent vertices whose degree sum is not less than n. Thus, $Cl(G+uv)$ is complete graph K_n, and by Corollary 1, $G+uv$ is Hamiltonia graph. This proves that, G contains a Hamiltonian path.

Note that, for $n=5$, $G(5,2)$ has only one graph as shown in Figure 3.

Suppose that, $G\in G(n,k)$, u and v are two non-adjacent vertices in G. Denote by N_v the set of vertices that are non-adjacent to v, N_u the set of vertices that are non-adjacent to u in G. Thus, $Z=V\setminus N_u\cup N_v$ is a set of vertices which are both adjacent to v and u, $A=N_u\cap N_v$ is a set of vertices which are non-adjacent to v and u.

Proposition 3. *For every* $G\in G(n,k)$*,* $|Z|=|A|+1$.

Proof. By all vertices of the G have degrees $(n-1)/2$, $|N(u)|=n-1-\deg(u)==n-1-(n-1)/2=(n-1)/2$. Similarly, $|N(v)|=(n-1)/2$. We have, $|Z|=|V|\setminus|N_u\cup N_v|=n-[|N_u|+|N_v|-|N_u\cap N_v|]=n-[(n-1)/2+(n-1)/2-|A|]=|A|+1$. Thus, $|Z|=|A|+1$.

Proposition 4. *For every* $G\in G(n,k)$*,* $d(G)=2$.

Proof. Let u and v be two non-adjacent vertices in G. By Proposition 3, $|Z|=|A|+1$, so $|Z|\geq 1$, or $Z\neq\varnothing$. This proves that, with two non-adjacent vertices u and v in G, there exists at least one vertex $z\in Z$ such that z is adjacent to both vertices u and v. In other words, $\forall(u,v)\notin E(G)$, $d(u,v)=2$. Thus, $d(G)=2$.

Proposition 5. *Let* $n\geq 9$*, for every* $G\in G(n,k)$*,* $3\leq\alpha(G)\leq(n-1)/2$.

Proof. a) Fisrt, we will prove that $3\leq\alpha(G)$.

Assume that $\alpha(G)=2$. Let u and v be two any non-adjacent vertices in G.

Consider 1. By $\alpha(G)=2$, so $A=\varnothing$, and by Proposition 3, $|Z|=1$. Let $Z=\{z\}$, and so z is the only vertex that is adjacent to both vertices u and v in G. Let Nuz be the set of vertices of N_u that are non-adjacent to z, Nvz be the set of vertices of N_v that are non-adjacent to z. Figure 4 illustrates a graph in $G(9,4)$ to prove Proposition 5.

Figure 4.

Obviously, $|Nvz|+|Nuz|=(n-1)/2$. Moreover, by $\alpha(G)=2$, each pair of vertices in Nuz must be adjacent, and each vertex in Nuz must be adjacent to every vertex in Nvz. Similarly, each pair of vertices in Nvz must be adjacent. In other words, the vertices in Nuz form a clique $K_{|Nuz|-1}$ and the vertices in Nvz form a clique $K_{|Nvz|-1}$ in G.

Consider 2. Suppose that w is any vertex in Nvz. Then, there exists at least one vertex $r\in N_v\setminus N_{vz}$ such that w is adjacent to r (if not, graph G will have three vertices w, r, v, where each pair is non-adjacent, is contradictory to hypothesis $\alpha(G)=2$).

From *Consider 1* and *Consider* 2, we have, vertex w must be adiacent to u, r and all vertices in Nuz and Nvz. I.e., $\deg(w)\geq 1+1+|Nvz|-1+|Nuz|=1+(n-1)/2$. This is contrary to the assumption of the k - regular graph G, $k=(n-1)/2$. So, $\alpha(G)\geq 3$.

b) Next, we will prove that $\alpha(G)\leq(n-1)/2$.

Assume that $\alpha(G)=(n+1)/2$, and let $S=\{s_1,s_2,...,s_{(n+1)/2}\}$ is a maximum independent set of G. Set $M=V\setminus S$. We have, $|M|=n-|S|=n-(n+1)/2=(n-1)/2$. For every $i\in\{1,2,...,(n+1)/2\}$, $\deg(s_i)=(n-1)/2$. So s_i is adjacent to $(n-1)/2$ vertices in M. I.e., each vertex in M must be adjacent to every vertex in $S=\{s_1,s_2,...,s_{(n+1)/2}\}$. This proves that, each vertex in M has degree no less than $(n+1)/2$, this is contrary to the assumption of the k-regular graph G. Therefore, $\alpha(G) \le (n-1)/2$.

Note that, Proposition 5 is also true for $n=5$, in $G(5,2)$ has the only graph G for $\alpha(G) = (5-1)/2 = 2$ (see Figure 3). Figure 5 illustrates graphs in $G(9,4)$ for $\alpha(G)=3$ and $\alpha(G)=4$.

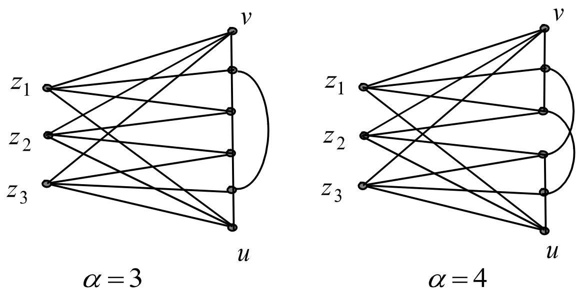

Figure 5.

Theorem 5. *Let $n \ge 9$, for every $G \in G(n,k)$, G is Hamiltonian graph.*

Proof. We will show that graph G satisfies the condition of Theorem 4, and therefore, Theorem 5 is proved.

Indeed, by $\delta(G) = k = (n-1)/2$ (the hypothesis of G) and $3 \le \alpha(G) \le (n-1)/2$ (Proposition 3), so $\alpha(G) \le \delta(G)$. (1)

By $n \ge 9$, we have $(n-1)/2 \ge (n+2)/3$, i.e., $\delta(G) \ge (n+2)/3$. (2)

Next, we show that $\kappa \ge 2$. Suppose othewise, $\kappa = 1$ and w is an any cut vertex of G. Then, graph $G-w$ is disconned graph, and in $G-w$ there exist two disjoint sets X and Y such that $V = \{w\} \cup X \cup Y$, $X \cap Y = \varnothing$. By, each vertex in G has degree $\delta = (n-1)/2$, so $|X| = |Y| = (n-1)/2$, all vertices of X (similarly Y) whose each pairwise are adjacent, and all vertices of $X \cup Y$ are adjacent to w. So, $\deg(w) = |X| + |Y| = (n-1)/2 + (n-1)/2 = n-1$, a contradiction with the hypothesis of G. Thus, $\kappa \ge 2$. (3)

From (1), (2), (3) shown that graph G satisfies the condition of Theorem 4.

REFERENCES

[1] An D. Nhu, “Recognizing the Hamiltonian graph with is an easy problem”, International Journal of Advanced Reseach in Computer Science, 10 (2), March- April 2019, 42-45.

[2] An D. Nhu, Duong B. Duc, “Some problems about Hamiltonian Cycle in special graphs”, The 2nd International Conference on Theories and Applications of Computer Science (ICTACS’09), Journal of Sciencce and Technology, Vietnam Academy of Science and Technology, Vol. 46, No 5A (2008), 57-66.

[3] Bondy J. A. and Chvátal V., A Method in Graph Theory, Discrete Math. 15, p. 111-135, 1976.

[4] Graham R., Lovász L. and Grotschel M., Handbook of combinatorics, Vol 1, Elsevier Science B.V., 1995.

[5] Paul Erdos, “A class of Hamiltonian regular graphs”, Journal of Graph Theory, Vol 2 (1978) 129-135.\

