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SOME FIXED POINT AND COMMON FIXED POINT RESULTS IN L- SPACES

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Abstract: In the present paper we have investigated fixed point and common fixed point theorems in ***L***-spaces as many other mathematicians Yeh [13], Singh [10], Pachpatte [6], Pathak and Dubey [7], Patel, Sahu and Sao [8], Patel and Patel [9], Som [11], Sao [12] worked for L-spaces.

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KeyWords: Fixed point, Common fixed point, Separated L- space.

1. INTRODUCTION AND PRILIMINARIES

It was shown by S. Kashara [4] & [5] in 1975 that several known generalization of the Banach contraction theorem can be derived easily from a fixed point theorem in an L-space. Iseki [2] has used the fundamental idea of Kashara to investigate the generalization of some known fixed point theorem in $L-$space.

Many other mathematicians Yeh [13], Singh [10], Pachpatte [6], Pathak and Dubey [7], Jain, R. K and Sahu, H. K [3], Patel, Sahu and Sao [8], Patel and Patel [9], Som [11], Sao [12] did lot of work in $\boldsymbol{L-}$**spaces**. Recently Bhardwaj, Rajpoot and Yadava [1] also worked on ***L*-spaces.** and produced some fixed point and common fixed point theorems. In this present paper a similar investigation for the study of fixed point and common fixed point theorems in $\boldsymbol{L}$ **−spaces** are worked out. We find some fixed point and common fixed point theorems in $\boldsymbol{L}$ **−spaces.**

Definition 1.1: $\boldsymbol{L-Space}$: Let N be a set of all non negative integers and X is a non-empty set. A pair $(X,\to)$ of a set X and a subset $\to$ of the set $X^N \times X$, is called an $L-space$ if

(i) If $x_n = x$, where $x \in X$, for all $n \in N$, then $(\{x_n\}_{n\in N}, x) \in \to$

(ii) $(\{x_n\}_{n\in N}, x) \in \to$, then $(\{x_{ni}\}_{i\in N}, x) \in \to$, for every $\{x_{n_i}\}_{i\in N}$ of $\{x_n\}_{n\in N}$

in what follows, we shall write $\{x_n\}_{n\in N} \to x$, Or $x_n \to x$ instead of $(\{x_n\}_{n\in N}, x) \in \to$ and read $\{x_n\}_{n\in N}$ *converges to x* .

Definition 1.2: An $L-$space $(X,\to)$ is said to be separated if each sequence in X converges to at most one point of X.

Definition 1.3: A mapping T of an $L-space$ $(X,\to)$ into an $L-$space $(X,\to)$ is said to be continuous if $x_n \to x \Rightarrow Tx_n \to Tx$, for some sub sequence $\{x_{n_i}\}_{i\in N}$, $\{x_n\}_{n\in N}$.

Definition 1. 4: Let d be non-negative extended real valued function on $X \times X$, $0 \le d(x,y) < \infty$ *for all* $x, y \in X$, an $L-$space $(X,\to)$ is said to be $d-$complete if each sequence, $\{x_n\}_{n\in N}$, in X with $\sum d(x_i, y_{i+1}) < \infty$ converges to at most one point of X.

Definition 1.5: Let $(X,\to)$ be an $L-$space which is $d-$complete for a non-negative real valued function d on $X \times X$, if $(X,\to)$ is separated, then $d(x,y) = d(y,x) = 0$, implies $x = y$ for every x, y *in* X

2. Main Results

Theorem 2.1: Let $(X,\to)$ be a separated $L-space$ which is $d-$complete for a non-negative real valued function d on $X \times X$, with $d(x,x) = 0$ for all x *in* X. Let E be a continuous self map of X satisfying the conditions:

$$[d(Ex,Ey)]^2 \le \emptyset\begin{Bmatrix} d(x,Ex)d(y,Ey), d(x,Ex)d(y,Ex), \\ d(y,Ey)d(y,Ex), d(x,Ey)d(y,Ex) \end{Bmatrix}$$

2.1(a)

$\forall\, x, y \in X$. Then E has a unique fixed point.

Proof: Let x_0 be an arbitrary point in X, define sequence $\{x_n\}$ recurrently,

$Ex_0 = x_1, Ex_1 = x_2, \dots \dots \dots \dots \dots . . Ex_n = x_{n+1}$ where, $n = 0,1,2,3, \dots$

Now by 2.1(a) we have

$$[d(x_1,x_2)]^2 = [d(Ex_0,Ex_1)]^2$$
$$\le \emptyset\begin{Bmatrix} d(x_0,Ex_0)d(x_1,Ex_1), d(x_0,Ex_0), d(x_1,Ex_0), \\ d(x_1,Ex_1)d(x_1,Ex_0), d(x_0,Ex_1)d(x_1,Ex_0) \end{Bmatrix}$$
$$\le$$
$$\emptyset\begin{Bmatrix} d(x_0,x_1)d(x_1,x_2), d(x_0,x_1), d(x_1,x_1), \\ d(x_1,x_2)d(x_1,x_1), d(x_0,x_2)d(x_1,x_1) \end{Bmatrix}$$
$$\le$$
$$h.d(x_0,x_1)d(x_1,x_2)$$

$$d(x_1,x_2) \le h.d(x_0,x_1)$$

Similarly

$$d(x_2,x_3) \le h.d(x_1,x_2)$$
$$\le h.h.d(x_0,x_1)$$

$$d(x_n,x_{n+1}) \le h^n.d(x_0,x_1),$$

where $h < 1$

for every natural number we can say that $\sum d(x_n,x_{n+1}) \le \infty$

By $d-completeness$ of the space, the sequence $\{E^n x_0\}, n \in N$ converges to some u in X. By continuity of E, the sub sequence $\{E^{ni}x_0\}$ also converges to u.

$$\lim_{i\to\infty} E^{ni+1}x_0 = E_u$$

$$\lim_{i\to\infty} E^{ni}x_0 = u$$

$$\mathrm{E}\,(\lim_{i\to\infty} E^{ni}x_0) = Eu$$

$$\lim_{i\to\infty} E^{ni+1}x_0 = Eu$$

$\Rightarrow E_u = u$, so u is a fixed point of E.

Uniqueness: In order to prove that u is the unique fixed point of E, if possible let v be any other fixed point of E ($v \neq u$). Then

$$d(u,v) = d(Eu,Ev)$$

$$[d(Eu,Ev)]^2 \leq \emptyset\begin{Bmatrix} d(u,Eu)d(v,Ev), d(u,Eu)d(v,Eu), \\ d(v,Ev)d(v,Eu), d(u,Ev)d(v,Eu) \end{Bmatrix}$$

$$[d(u,v)]^2 \leq h[d(u,v)]^2$$

This is a contradiction because $h < 1$. So E has a unique fixed point in X.

Now we will prove another fixed point theorem which is stronger than **theorem 2.1.**

Theorem 2.2: Let $(X, \rightarrow)$ be a separated L −space which is d −complete for a non-negative real valued function d on $X \times X$, with $d(x,x) = 0$ for all $x \, in \, X$. Let E be a continuous self map of X satisfying the conditions:

$$d(Ex,Ey) \leq max\begin{Bmatrix} d(x,Ex)d(y,Ey), d(x,Ex)d(y,Ex), \\ d(y,Ey)d(y,Ex), d(x,Ey)d(y,Ex) \end{Bmatrix}^{\frac{1}{2}} \qquad 2.2(a)$$

$\forall \, x,y \in X$. Then E has a unique fixed point.

Proof: Let x_0 be an arbitrary point in X, define sequence $\{x_n\}$ recurrently,

$$Ex_0 = x_1, Ex_1 = x_2, \dots \dots \dots \dots Ex_n = x_{n+1}$$

where $n = 0,1,2,3, \dots$

Now by 2.2(a) we have

$$d(x_1,x_2) = d(Ex_0,Ex_1) \leq$$

$$max\begin{Bmatrix} d(x_0,Ex_0)d(x_1,Ex_1), d(x_0,Ex_0)d(x_1,Ex_0), \\ d(x_1,Ex_1)d(x_1,Ex_0), d(x_0,Ex_1)d(x_1,Ex_0) \end{Bmatrix}^{\frac{1}{2}} \leq$$

$$max\begin{Bmatrix} d(x_0,x_1)d(x_1,x_2), d(x_0,x_1)d(x_1,x_1), \\ d(x_1,x_2)d(x_1,x_1), d(x_0,x_2)d(x_1,x_1) \end{Bmatrix}^{\frac{1}{2}} \leq$$

$$\{d(x_0,x_1)d(x_1,x_2)\}^{\frac{1}{2}}$$

$$d(x_1,x_2) \leq d(x_0,x_1)$$

Similarly $\quad d(x_2,x_3) \leq d(x_1,x_2) \leq d(x_0,x_1)$

$$d(x_n,x_{n+1}) \leq \dots \dots \dots \leq d(x_0,x_1),$$

for every natural number we can say that

$$\sum d(x_n,x_{n+1}) \leq \infty$$

By d −completeness of the space, the sequence $\{E^n x_0\}$, $n \in N$ converges to some u in X. By continuity of E, the sub sequence $\{E^{ni} x_0\}$ also converges to u.

$$\lim_{i\to\infty} E^{ni+1}x_0 = E_u$$

$$\lim_{i\to\infty} E^{ni}x_0 = u$$

$$E(\lim_{i\to\infty} E^{ni}x_0) = Eu$$

$$\lim_{i\to\infty} E^{ni+1}x_0 = Eu$$

$\Rightarrow E_u = u$, so u is a fixed point of E.

Uniqueness: Now to prove the uniqueness of the fixed point u of E, contrarily assume that there is another possible fixed point v of E and $v \neq u$. Then

$$d(u,v) = d(Eu,Ev)$$

$$d(Eu,Ev) \leq \begin{Bmatrix} d(u,Eu)d(v,Ev), d(u,Eu)d(v,Eu), \\ d(v,Ev)d(v,Eu), d(u,Ev)d(v,Eu) \end{Bmatrix}^{\frac{1}{2}}$$

$$d(u,v) \leq d(u,v)$$

This is a contradiction. So E has a unique fixed point in X.

Theorem 2.3: Let $(X, \rightarrow)$ be a separated L −space which is $d - complete$ for a non-negative real valued function d on $X \times X$, with $d(x,x) = 0$ for all $x \, in \, X$. Let E and T be two continuous self map of X satisfying the conditions:

$$ET = TE, \; E(X) \subseteq T(X)$$

2.3(a)

$$[d(Ex,Ey)]^2 \leq \emptyset\begin{Bmatrix} d(Tx,Ex)d(Ty,Ey), d(Tx,Ex)d(Ty,Ex), \\ d(Ty,Ey)d(Ty,Ex), d(Tx,Ey)d(Ty,Ex) \end{Bmatrix}$$

2.3(b)

$\forall \, x,y \in X$. Then E and T has a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X, since $E(X) \subseteq T(X)$**,** we can choose $x_1 \in X$ such that $Ex_0 = Tx_1$, $Ex_1 = Tx_2, \dots \dots \dots \dots \dots Ex_n = Tx_{n+1}$ For $n = 1,2,3, \dots$

$$[d(Tx_{n+1},Tx_{n+2})]^2 = [d(Ex_n,Ex_{n+1})]^2$$

$$\leq \emptyset\begin{Bmatrix} d(Tx_n,Ex_n)d(Tx_{n+1},Ex_{n+1}), d(Tx_n,Ex_n)d(Tx_{n+1},Ex_n), \\ d(Tx_{n+1},Ex_{n+1})d(Tx_{n+1},Ex_n), d(Tx_n,Ex_{n+1})d(Tx_{n+1},Ex_n) \end{Bmatrix} \leq$$

$$\emptyset\begin{Bmatrix} d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+2}), d(Tx_n,Tx_{n+1}), d(Tx_{n+1},Tx_{n+1}), \\ d(Tx_{n+1},Tx_{n+2})d(Tx_{n+1},Tx_{n+1}), d(Tx_n,Ex_{n+1})d(Tx_{n+1},Tx_{n+1}) \end{Bmatrix}$$

$$d(Tx_{n+1},Tx_{n+2}) \leq h.\, d(Tx_n,Tx_{n+1})$$

Hence, $d(Tx_{n+1},Tx_{n+2}) \leq h^n.\, d(Tx_1,Tx_2)$

For every natural number m ,we can write the $\sum_m^\infty d(x_m,x_{m+1}) < \infty$

By $d - completeness$ of x, the sequence $\{T^n x_0\}_{n\in N}$ converges to some $u \in X$.Since $E(x) \subseteq T(x)$, therefore the subsequence t of $\{T^n x_0\}$ such that, $E(T(u)) \rightarrow Eu$, and $T(E(u)) \rightarrow Tu$,So we have, $Eu = Tu$

Since $\lim_{n\to\infty} T^n X_0 = u$, $T(\lim_{n\to\infty} T^n X_0) = Tu$---------------------(2.3.1)

This implies that $Tu = u$. Hence $Tu = Eu = u$

Thus u is common fixed point of E and T.

Uniqueness: For the uniqueness of the common fixed point, if possible let v be any other common fixed point of $E$and T. Then from 2.3(b)

$$d(u,v) = d(Eu,Ev)$$

$$[d(Eu,Ev)]^2 \leq \emptyset\begin{Bmatrix} d(Tu,Eu)d(Tv,Ev), d(Tu,Eu)d(Tv,Eu), \\ d(Tv,Ev)d(Tv,Eu), d(Tu,Ev)d(Tv,Eu) \end{Bmatrix}$$

$$[d(u,v)]^2 \leq h.[d(u,v)]^2$$

Which is a contradiction because $h < 1$. Hence E and T have a unique common fixed point in X.

Theorem 2.4: Let$(X, \rightarrow)$be a separated L-Space which is d-complete for a non negative real valued function d on $X \times X$ with $d(x,x) = 0$ for all x in X. Let E and T be two continuous self mappings of X. Satisfying the conditions:

$$ET = TE, E(X) \subseteq T(X) \qquad 2.4(a)$$

$$d(Ex,Ey) \leq max\begin{Bmatrix} d(Tx,Ex)d(Ty,Ey), d(Tx,Ex)d(Ty,Ex), \\ d(Ty,Ey)d(Ty,Ex), d(Tx,Ey)d(Ty,Ex) \end{Bmatrix}^{\frac{1}{2}}$$

2.4(b)

$\forall \, x,y \in X$. Then E and T has a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X, since $E(X) \subseteq T(X)$**,** we can choose $x_1 \in X$ such that $Ex_0 = Tx_1$, $Ex_1 = Tx_2, \dots \dots \dots \dots \dots Ex_n = Tx_{n+1}$. For $n = 1,2,3, \dots$

$$d(Tx_{n+1},Tx_{n+2}) = d(Ex_n,Ex_{n+1})$$

$$\leq max\begin{Bmatrix} d(Tx_n,Ex_n)d(Tx_{n+1},Ex_{n+1}), d(Tx_n,Ex_n)d(Tx_{n+1},Ex_n), \\ d(Tx_{n+1},Ex_{n+1})d(Tx_{n+1},Ex_n), d(Tx_n,Ex_{n+1})d(Tx_{n+1},Ex_n) \end{Bmatrix}^{\frac{1}{2}}$$

$$\leq \max\begin{Bmatrix} d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+2}), d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+1}), \\ d(Tx_{n+1},Tx_{n+2})d(Tx_{n+1},Tx_{n+1}), d(Tx_n,Tx_{n+2})d(Tx_{n+1},Tx_{n+1}) \end{Bmatrix}^{\frac{1}{2}}$$

$$\leq \{d(Tx_n,Tx_{n+1})\}^{\frac{1}{2}}$$

$$d(Tx_{n+1},Tx_{n+2}) \leq d(Tx_1,Tx_2),$$

for every natural number m, we can say that $\sum_m^\infty d(x_m,x_{m+1}) < \infty$

By d −completeness of X, the sequence $\{T^n x_0\}_{n\in N}$, converges to some u in X. Since $E(X \subseteq T(X)$,so $E(T(u)) \to Eu$,and $T(E(u)) \to Tu$ we have $Eu \to Tu$,

Since $\lim_{n\to\infty} T^n x_0 = u$

$$T(\lim_{n\to\infty} T^n x_0) = Tu \quad \text{-----------------------------(2.4.1)}$$

$\Rightarrow Tu = u$,

Hence $Tu = Eu = u$ .So u is common fixed point of E *and* T.

Uniqueness: In order to prove that u is the unique fixed point of E, if possible let v be any other fixed point of E *and* T, $(v \neq u)$. Then from (2.4) (b)

$$d(u,v) = d(Eu,Ev)$$

$$d(Eu,Ev) \leq \max\begin{Bmatrix} d(Tu,Eu)d(Tv,Ev), d(Tu,Eu)d(Tv,Eu), \\ d(Tv,Ev)d(Tv,Eu), d(Tu,Ev)d(Tv,Eu) \end{Bmatrix}^{\frac{1}{2}}$$

$$d(u,v) \leq [d(u,v)]^{\frac{1}{2}}$$

Which is a contradiction. Hence E *and* T have a unique common fixed point in X.

In next theorems we will prove the common fixed point theorems for three mappings.

Theorem 2.5: Let$(X, \to)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x) = 0$ for all x in X. Let E, F, T be three continuous self mappings of X. Satisfying the conditions:

$$ET = TE, FT = TF, E(X) \subset T(X) And\ F(X) \subset T(X) \qquad 2.5(a)$$

$$[d(Ex,Fy)]^2 \leq \emptyset\begin{Bmatrix} d(Tx,Ex)d(Ty,Fy), d(Tx,Ex)d(Ty,Ex), \\ d(Ty,Fy)d(Ty,Ex), d(Tx,Fy)d(Ty,Ex) \end{Bmatrix} \qquad 2.5(b)$$

$\forall\, x,y \in X$. Then E, F, T has a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X, since $E(X) \subset T(X)$**,**we can choose $x_1 \in X$ such that $Tx_1 = Ex_0$,also $F(x) \subset T(X)$.We can choose a point x_2 in X such that $Tx_2 = Fx_1$.

In general we can choose the point

$Tx_{2n+1} = Ex_{2n}$, -----------------(2.5.1)

$Tx_{2n+2} = Fx_{2n+1}$, --------------(2.5.2)

For every $n \in N$, we have

$$[d(Tx_{2n+1},Tx_{2n+2})]^2 = [d(Ex_{2n},Fx_{2n+1})]^2$$

$$[d(Ex,Fy)]^2 \leq \emptyset\begin{Bmatrix} d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Fx_{2n+1}), d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Ex_{2n}), \\ d(Tx_{2n+1},Fx_{2n+1})d(Tx_{2n+1},Ex_{2n}), d(Tx_{2n},Fx_{2n+1})d(Tx_{2n+1},Ex_{2n}) \end{Bmatrix}$$

$d(Tx_{2n+1},Tx_{2n+2}) \leq h\, d(Tx_{2n},Tx_{2n+1})$ For $n = 1,2,3,$ ------------------,

Similarly

$$(Tx_{2n+1},Tx_{2n+2}) \leq h^n d(Tx_1,Tx_0)$$

$$\sum_{i=0}^\infty d(Tx_{2i+1},Tx_{2i+2}) < \infty$$

Thus the d −completeness of the space implies the sequence $\{T^n x_0\}_{n\in N}$ converges to some $u \in X$. So by (2.5.1) and (2.5.2), $(E^n x_0) n \in N$, and $(F^n x_0) n \in N$ also converges to the some point u respectively.

Since $E, T and\ F$ are continuous, there is a subsequence t of $\{T^n x_0\}, n \in N$ such that $E(T(t)) \to Eu, T(E(t)) \to Tu, F(T(t)) \to Fu$ and $T(F(t)) \to Tu$

Hence we have, $Eu = Fu = Tu$ ---(2.5.3)

Thus $T(Tu) = T(Eu) = E(Fu) = T(Fu) = F(Tu) = E(Tu) = F(Eu) = F(Fu)$---------(2.5.4)

Now if $Eu \neq F(Eu)$

$$[d(Eu,F(Eu))]^2 \leq \emptyset\begin{Bmatrix} d(Tu,Eu)d(T(Eu),F(Eu)), d(Tu,Eu)d(T(Eu),Eu), \\ d(T(Eu),F(Eu))d(T(Eu),Eu), d(Tu,F(Eu))d(T(Eu),Eu) \end{Bmatrix}$$

$$[d(Eu,F(Eu))]^2 \leq h\, d\,(Tu,F(Eu))d\,(Tu,F(Eu))$$

$$[d(Eu,F(Eu))] \leq h[d(Eu,F(Eu))]$$

Which is a contradiction. Hence $Eu = F(Eu)$ ---(2.5.5)

From (2.5.4) and (2.5.5) we have

$$Eu = F(Eu) = T(Eu) = E(Eu)$$

Hence Eu is a common fixed point of $E, F and\ T$.

Uniqueness: For the uniqueness of the common fixed point, if possible let u and v ,$(v \neq u)$ be two common fixed point of E, F and T.Then from (2.5) (b) $d(u,v) = d(Eu,Fv)$

$$[d(Eu,Fv)]^2 \leq \emptyset\begin{Bmatrix} \alpha d(Tu,Eu)d(Tv,Fv), \beta d(Tu,Eu)d(Tv,Eu), \\ d(Tv,Fv)d(Tv,Eu), d(Tu,Fv)d(Tv,Eu) \end{Bmatrix}$$

$$[d(u,v)]^2 \leq h.[d(u,v)]^2$$

Which is a contradiction. Hence $u = v$.

So $E, F and\ T$ have a unique common fixed point.

Theorem 2.6: Let $(X, \to)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x) = 0$ for all x in X. Let $E, F and\ T$ be three continuous self mappings of X. Satisfying the conditions:

$$ET = TE, FT = TF, E(X) \subset T(X) And\ F(X) \subset T(X) \qquad 2.6(a)$$

$$d(Ex,Fy) \leq \max\begin{Bmatrix} d(Tx,Ex)d(Ty,Fy), d(Tx,Ex)d(Ty,Ex), \\ d(Ty,Fy)d(Ty,Ex), d(Tx,Fy)d(Ty,Ex) \end{Bmatrix}^{\frac{1}{2}} \qquad 2.6(b)$$

$\forall\, x,y \in X$. Then E, F *and* T has a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X, since $E(X) \subset T(X)$**,** we can choose $x_1 \in X$ such that $Tx_1 = Ex_0$,also $F(x) \subset T(X)$.We can choose a point x_2 in X such that $Tx_2 = Fx_1$.

In general we can choose the point

$Tx_{2n+1} = Ex_{2n}$, -----------------(2.6.1)

$Tx_{2n+2} = Fx_{2n+1}$, --------------(2.6.2)

For every $n \in N$, we have

$$d(Tx_{2n+1},Tx_{2n+2}) = d(Ex_{2n},Ex_{2n+1})$$

$$\leq \max\begin{Bmatrix} d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Fx_{2n+1}), d(Tx_n,Ex_n)d(Tx_{2n+1} \ldots \\ d(Tx_{2n+1},Fx_{2n+1})d(Tx_{2n+1},Ex_{2n}), d(Tx_{2n},Fx_{2n+1})d(Tx \ldots \end{Bmatrix}$$

$d(Tx_{2n+1},Tx_{2n+2}) \leq d(Tx_{2n},Tx_{2n+1})$ *For* $n =$ 1,2,3,-------------------

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$$d(Tx_{2n+1},Tx_{2n+2}) \leq d(Tx_1,Tx_0),$$

$$\sum_{i=0}^{\infty} d(Tx_{2i+1}, Tx_{2i+2}) < \infty$$

Thus the $d-completeness$ of the space implies the sequence $\{T^n x_0\}_{n\in N}$ converges to some $u \in X$. So $(E^n x_0) n \in N$, and $(F^n x_0) n \in N$ also converges to the some point u respectively.

Since $E, T and\ F$ are continuous, there is a subsequence t of $\{T^n x_0\}, n \in N$ such that $E(T(t)) \to Eu, T(E(t)) \to Tu, F(T(t)) \to Fu$ and $T(F(t)) \to Tu$

Hence we have $Eu = Fu = Tu$ ---(2.6.3)

Thus $T(Tu) = T(Eu) = E(Fu) = T(Fu) = F(Tu) = E(Tu) = F(Eu) = F(Fu)$-------(2.6.4)

Now if $Eu \neq F(Eu)$

$d(Eu, F(Eu)) \leq$

$$\left\{\begin{matrix} d(Tu,Eu)d(T(Eu),F(Eu)), d(Tu,Eu)d(T(Eu),Eu), \\ d(T(Eu),F(Eu))d(T(Eu),Eu), d(Tu,F(Eu))d(T(Eu),Eu) \end{matrix}\right\}^{\frac{1}{2}}$$

$d(Eu, F(Eu)) \leq$

$\{[d(Tu, F(Eu))d(Tu, F(Eu))]\}^{\frac{1}{2}}$

$d(Eu, F(Eu))] \leq [d(Eu, F(Eu))]$

Which is a contradiction. Hence $Eu = F(Eu)$ ---(2.6.5)

so $Eu = F(Eu) = T(Eu) = E(Eu)$

Hence Eu is a common fixed point of $E, F\ and\ T$.

Uniqueness: In order to prove that u is the unique common fixed point of $E, F$and T. If possible let v be any other common fixed point of $F\ and\ T$ $(v \neq u)$. Then we have

$$d(u, v) = d(Eu, Fv)$$

$d(Eu, Fv) \leq$

$$max\left\{\begin{matrix} d(Tu,Eu)d(Tv,Fv), d(Tu,Eu)d(Tv,Eu), \\ d(Tv,Fv)d(Tv,Eu), d(Tu,Fv)d(Tv,Eu) \end{matrix}\right\}^{\frac{1}{2}}$$

$d(u, v) \leq d(u, v)$

Which is a contradiction. Hence $u = v$. So $E, F\ and\ T$ have a unique common fixed point in .

Theorem 2.7: Let $(X, \rightarrow)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x, x) = 0$ for all x in X. Let $E, Fand\ T$ be three continuous self mappings of X. satisfying the conditions:

$ET = TE, FT = TF, E(X) \subset T(X)\ and\ F(X) \subset T(X)$ 2.7(a)

$[d(E^p x, F^q y)]^2 \leq$

$$\emptyset\left\{\begin{matrix} d(Tx,E^p x)d(Ty,F^q y), d(Tx,E^p x)d(Ty,E^p x), \\ d(Ty,F^q y)d(Ty,E^p x), d(Tx,F^q y)d(Ty,E^p x) \end{matrix}\right\}$$

2.7(b)

$\forall\, x, y \in X$. $Tx \neq Ty$. If some positive integer p, q exists such that $E^p, F^q and\ T$ are continuous. Then $E, F\ and\ T$ have a unique fixed point in X.

Proof: It follows from 2.6(a)

$E^p T = TE^p, F^q T = TF^q, E^p(X) \subset T(X)$-----------------(2.7.1)

and $F^q(X) \subset T(X)$------------------(2.7.2)

i.e. u is the fixed point of $T, E^p and\ F^q$.

Now$T(Eu) = E(Tu) = E(u) = E(E^p u) = E^p(Eu)$------(2.7.3) $T(Fu) = F(Tu) = F(u) = F(F^q u) = F^q(Fu)$------(2.7.4)

Hence it follows that Eu is common fixed point of $T, E^p\ and\ Fu$ is a common fixed point of $T\ and\ F^q$. The uniqueness of u, can be proved easily.

Theorem 2.8: Let $(X, \rightarrow)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x, x) = 0$ for all x in X. Let $E, Fand\ T$ be three continuous self mappings of X, satisfying the conditions:

$d(E^p x, F^q y) \leq$

$$\emptyset\left\{\begin{matrix} d(Tx,E^p x)d(Ty,F^q y), d(Tx,E^p x)d(Ty,E^p x), \\ d(Ty,F^q y)d(Ty,E^p x), d(Tx,F^q y)d(Ty,E^p x) \end{matrix}\right\}^{\frac{1}{2}}$$

2.8(a)

$\forall\, x, y \in X$. $Tx \neq Ty$ .If some positive integer p, q exists such that $E^p, F^q and\ T$ are continuous. Then E, F, T have a unique fixed point in X.

Proof: The proof is similar as the Theorem (2.7).

Now we will prove some common fixed point theorem for four mappings, which contains rational expressions.

Theorem 2.9: Let $(X, \rightarrow)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x, x) = 0$ for all x in X. Let $E, F, Tand\ S$ be four continuous self mappings of X. Satisfying the conditions:

$ES = SE, FT = TF, E(X) \subset T(X)\ and\ F(X) \subset S(X)$ 2.9(a)

$d(Ex, Fy) \leq$

$$\alpha\ max\left[d(Sx,Ty), \left\{\frac{d(Ex,Ty)+d(Sx,Fx)}{d(Sx,Ty)+d(Ex,Ty)}\right\}, d(Sx,Ex), \left\{\frac{d(Fy,Ty)+d(Ex,Ty)}{d(Sx,Ty)+d(Ex,Ty)}\right\}\right]$$

2.9(b)

$$\forall\, x, y \in X\ with\ [d(Sx, Ty) + d(Ex, Ty)] \neq 0.$$

Then $E, F, T\ and\ S$ have a unique fixed point.

Proof: Let $x_0 \in X$, there exists a point $x_1 \in X$,such that $Tx_1 = Ex_0$ and for this point $x_1$we can choose a point $x_2 \in X$,such that $Fx_1 = Sx_2$ and so on inductively,we can define a sequence $\{y_n\}$ in X such that

$y_{2n} = Tx_{2n+1} = Ex_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Fx_{2n+1}$ where $n = 0,1,2, ---$

We have, $d(y_{2n}, y_{2n+1}) = d(Ex_{2n}, Fx_{2n+1})$

$$\leq \alpha\ max\left\{\begin{matrix} d(Sx_{2n},Tx_{2n+1}), \left\{\frac{d(Ex_{2n},Tx_{2n+1})+d(Sx_{2n},Fx_{2n})}{d(Sx_{2n},Tx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}\right\}, \\ d(Sx_{2n},Ex_{2n}), \left\{\frac{d(Fx_{2n+1},Tx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}\right\} \end{matrix}\right\}$$

$$\leq \alpha\ max\left\{\begin{matrix} d(Sx_{2n},Tx_{2n+1}), \left\{\frac{d(Tx_{2n+1},Tx_{2n+1})+d(Tx_{2n+1},Tx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Tx_{2n+1},Tx_{2n+1})}\right\}, \\ d(Sx_{2n},Tx_{2n+1}), \left\{\frac{d(Fx_{2n+1},Tx_{2n+1})+d(Tx_{2n+1},Tx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Tx_{2n+1},Tx_{2n+1})}\right\} \end{matrix}\right\}$$

$$\leq \alpha\ max\left\{\begin{matrix} d(Sx_{2n},Tx_{2n+1}), 0, \\ d(Sx_{2n},Tx_{2n+1}), \left\{\frac{d(Sx_{2n},Fx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})}\right\} \end{matrix}\right\}$$

Case I: $d(y_{2n}, y_{2n+1}) = d(Ex_{2n}, Fx_{2n+1}) \leq \alpha\,\{d(Sx_{2n}, Tx_{2n+1})\}$

Hence $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$

For every integer $p > 0$, we get

$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) +$ ------ $+d(y_{n+p-1}, y_{n+p})$

$$d(y_n, y_{n+p}) \leq \{\frac{a^p}{1-a}\} d(y_n, y_{n+1})$$

Letting $n \to \infty$, we have $d(y_n, y_{n+p}) \to 0$. Therefore $\{y_n\}$ is a Cauchy sequence in X. By $d-$completeness of X, $\{y_n\}_{n\in N}$ is converges to some $u \in X$. So subsequence $\{Ex_{2n}\}, \{Fx_{2n+1}\}, \{Tx_{2n+1}\} and \{Sx_{2n+2}\} of \{y_n\}$ also converges to same point u. Since $E, F, T\ and\ S$ are continuous, such that

$$E[S(x_n)] \to Eu, S[E(x_n)] \to Su, F[T(x_n)] \to Fu\ and\ T[F(x_n)] \to Tu$$

So, by weak compatibility $Eu = Su,\ Fu = Tu$. Now from 2.9(a) and 2.9(b) $d(E^2x_{2n}, Fx_{2n+1}) = d(E(Ex_{2n}), Fx_{2n+1})$

$$\leq \alpha\ max \begin{Bmatrix} d(S(Ex_{2n}), Tx_{2n+1}), \left\{\frac{d(E(Ex_{2n}),Tx_{2n+1})+d(S(Ex_{2n}),E(Ex_{2n}))}{d(S(Ex_{2n}),Tx_{2n+1})+d(E(Ex_{2n}),Tx_{2n+1})}\right\}, \\ d(S(Ex_{2n}), E(Ex_{2n})), \left\{\frac{d(Fx_{2n+1},Tx_{2n+1})+d(E(Ex_{2n}),Tx_{2n+1})}{d(S(Ex_{2n}),Tx_{2n+1})+d(E(Ex_{2n}),Tx_{2n+1})}\right\} \end{Bmatrix}$$

$$\leq \alpha\ max \begin{Bmatrix} d(Su, u), \left\{\frac{d(Eu,u)+d(Su,Eu)}{d(Su,u)+d(Eu,u)}\right\}, \\ d(Su, Eu), \left\{\frac{d(u,u)+d(Eu,u)}{d(Su,u)+d(Eu,u)}\right\} \end{Bmatrix}$$

$$d(Eu, u) \leq d(Su, u) = \alpha\ d(Eu, u)$$

This is a contradiction, because $\alpha < 1$.

So $Eu = Su = u$, that is u is common fixed point of $E\ and\ S$. Similarly we can prove $Fu = Tu = u$. So $E, F, T\ and\ S$ have common fixed point.

Uniqueness: In order to prove that u is the unique common fixed point of $E, F, T and\ S$. If possible let v be any other common fixed point of $, F, T\ and\ S\ \ (v \neq u)$. Then we have

$$d(u, v) = d(Eu, Fv)$$

$$d(Eu, Fv) \leq \alpha\ max \begin{Bmatrix} d(Su, Tv), \left\{\frac{d(Eu,Tv)+d(Su,Eu)}{d(Su,Tv)+d(Eu,Tv)}\right\}, \\ d(Su, Eu), \left\{\frac{d(Fv,Tv)+d(Eu,Tv)}{d(Su,Tv)+d(Eu,Tv)}\right\} \end{Bmatrix}$$

$$d(u, v) \leq \alpha d(u, v)$$

Which is a contradiction because < 1 . Hence u is the a unique common fixed point of $E, F, T\ and\ S$.

This completes the proof.

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