RESEARCH PAPER

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N- FOURIER SERIES EQUATIONS INVOLVING JACOBI POLYNOMIALS OF DIFFERENT INDICES

Indu Shukla
[1]Research scholar, Department of Physical science,
M.G.C.G.V.,Chitrakoot,
Satna, M.P.(India)

Uday singh
Department of Mathematics and Computer science,
R.D. University
,Jabalpur, M.P.(India)

Abstract: In this paper, we have considered the N-Fourier series equations involving Jacobi polynomials of different indices of the first and second kind and solved the two sets of series equations.

Keywords: Integral equation; Series equation; Fourier series; Integral theorems; Jacobi polynomials.

1. INTRODUCTION

Dwivedi and Trivedi [2] Considered quadruple series equations involving Jacobi polynomials of the same indices which are orthogonal to the weight function $(1-x)^{\alpha-\beta}x^{\beta-1}$ in the interval (0,1) Szego [9] later on standardized the notation $P_n^{(\alpha,\beta)}(x)$ for the Jacobi polynomials.

Askey [1] remarked that the dual series equations involving Jacobi polynomial of different indices cannot be solved. Later on Dwivedi and Gupta [4] obtained the solution of such quadruple series equations which include dual and triple series as particular cases. If we review the literature then we observe that the existing solutions on series equations are derived only from dual to six Fourier series equations, no further generalizations are available till date. This tempted us to find the solution of n-Fourier series equations involving Jacobi polynomials of different indices and in this paper we have obtained certain results. By considering the special values of n = 2,3,4,5,6 we shall be able to derive solutions of dual, triple,quadruple,5-tuple and 6-tuple Fourier series equations involvingjacobi polynomials of different indices[5], [6].

I.   N- SERIES EQUATIONS OF THE FIRST KIND

1) N-series equations of the first kind involving jacobi polynomials of different indices are as follows **:**

$$\sum_{m=0}^{\infty} \frac{A_m P_m^{(\alpha,\beta)}(x)}{(\beta+1)_m \ \Gamma(\alpha-\mu+m+1)} = f_i(x), \qquad a_{i-1} < x < a_i \tag{1}$$

where, $i = 1,3,5,\ldots,n-1$ and $a_0 = 0$.

$$\sum_{m=0}^{\infty} \frac{A_m P_m^{(\alpha+\beta-\delta, \ \delta)}(x)}{(\alpha+\beta-\delta+1)_m \ \Gamma(\beta+\mu+m+1)} = f_j(x), \ a_{i-1} < x < a_i \tag{2}$$

where, $j = 2,4,6,\ldots,n$ and $a_m = \infty$

Here n is taken as an even number. If n is odd then the equations will be

$$\sum_{m=0}^{\infty} \frac{B_m P_m^{(\alpha,\beta)}(x)}{(\beta+1)_m \ \Gamma(\alpha-\mu+m+1)} = f_i(x), \ a_{i-1} < x < a_i \tag{3}$$

where, $i = 1,3,5,\ldots,n$ and $a_0 = 0$.

$$\sum_{m=0}^{\infty} \frac{B_m P_m^{(\alpha+\beta-\delta, \ \delta)}(x)}{(\alpha+\beta-\delta+1)_m \ \Gamma(\beta+\mu+m+1)} = f_j(x), \ a_{j-1} < x < a_j \tag{4}$$

where, $j = 2,4,6,\ldots,n-1$.

2)   N-series equations of the second kind

N-series equations of the second kind involving jacobi polynomials of different indices are as follows :

$$\sum_{m=0}^{\infty} \frac{D_m P_m^{(\alpha+\beta-\delta, \ \delta)}(x)}{(\alpha+\beta-\delta+1)_m \ \Gamma(\beta+\mu+m+1)} = g_i(x), \ a_{i-1} < x < a_i \tag{5}$$

where, $i = 1,3,5,\ldots,n-1$ and $a_0 = o$

$$\sum_{m=0}^{\infty} \frac{D_m P_m^{(\alpha,\beta)}(x)}{(\beta+1)_m \ \Gamma(\alpha-\mu+m+1)} = g_j(x), \ a_{j-1} < x < a_j \tag{6}$$

where, $j = 2,4,6,\ldots,n$.

Here also n is taken as an even number. If n is odd then the equations will be

$$\sum_{m=0}^{\infty} \frac{E_m P_m^{(\alpha+\beta-\delta, \ \delta)}(x)}{(\alpha+\beta-\delta+1)_m \ \Gamma(\beta+\mu+m+1)} = g_i(x), a_{i-1} < x < a_i \tag{7}$$

where, $i = 1,3,5,\ldots,n$ and $a_0 = 0$.

$$\sum_{m=0}^{\infty} \frac{E_m P_m^{(\alpha,\beta)}(x)}{(\beta+1)_m \ \Gamma(\alpha-\mu+m+1)} = g_j(x), \ a_{j-1} < x < a_j \tag{8}$$

where, $j = 2,4,6,\ldots,n-1$.

Also $c > 0, 1$ is an arbitrary non- negative integer. $f_i(x), g_i(x)$, where i = 1,3,5,....,n-1 and $f_j(x), g_j(x)$ where $j = 2,4,6,\ldots,n$ are prescribed functions. A_m, B_m, D_m and E_m are unknown coefficients, are determined and the parameters $\alpha,\beta,\upsilon,\delta,\lambda$ satisfy the conditions $\delta - \mu - \beta > 0$, $\alpha > -1, \ \beta > -1, \delta > -1, \alpha + \beta - \delta > -1$. Here we solve only equations (1),(2)of first kind and equations (5),(6) of the first kind and equations (7),(8) of the second kind will follow easily.

2. PRELIMINARY RESULTS

In the course of analysis, we shall use the following results:
(i) The orthogonality relation for the Jacobi polynomials

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx =$$
$$\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\delta_n^m}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \tag{9}$$

where δ_n^m is the Kronecker delta,

(ii) The series ,

$$S(r,x) = \sum_{n=0}^{\infty} \frac{(\alpha+\beta-\delta+1)_n \, \Gamma(\beta+\mu+n+1)(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+\beta-\delta+1)\Gamma(n+\delta+1)(\beta+1)_n\Gamma(n+1+\alpha-\mu)}$$
$$\cdot P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) P_n^{(\alpha+\beta-\delta,\delta)}(r)$$
$$= a_n^{*}(1-r)^{-\delta}(1+x)^{-\beta}\int_{-1}^{w}\eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1}dy$$
$$= a_n^{*}(1-r)^{-\delta}(1+x)^{-\beta}S_w(r,x) \tag{10}$$

where, $w = \min(x,y)$ and $\eta(y) = (1+y)^{\delta-\mu}(1-y)^{\delta-\alpha+\beta}$

$$a_n^{*} = \frac{[\Gamma(n+1+\alpha)]^2\Gamma(\beta-\delta+\mu)\Gamma(1+\beta)\Gamma(1+\beta+\mu+n)}{\Gamma\mu\Gamma(\alpha+\beta-\delta+1)\Gamma(1+\delta-\mu+n)\Gamma(\alpha+\beta+\mu-\delta+n+1)} \tag{11}$$

It is assumed that parameters are so constrained that a_n^{*} is independent of n, this is of course possible when, for instance $\alpha = \nu$ = λ - μ` and $\beta = \delta$.

3. THE SOLUTION

(i) The solution of N-series equations of the first kind involving Jacobi polynomials of different indices :

Let us assume

$$\sum_{m=0}^{\infty} \frac{A_m P_m^{(\alpha+\beta-\delta,\ \delta)}(x)}{(\alpha+\beta-\delta+1)_m \, \Gamma(\beta+\mu+m+1)} = \phi_i(x), \quad a_{i-1} < x < a_i \tag{12}$$

where, $i = 2,4,6,\ldots,n.$

and where $\phi_i(x)$ are unspecified functions. Using orthogonality relation it follows from equations (1) and (12)

$$A_m = \frac{(\alpha+\beta-\delta+1)_m \, \Gamma(\beta+\mu+m+1)(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(m+\alpha+\beta-\delta+1)\Gamma(m+\delta+1)}$$
$$\sum_{i=0}^{\frac{n-2}{2}}\left\{\int_{a_{2i}}^{a_{2i+1}} f_{2i+1}(x) + \int_{a_{2i+1}}^{a_{2i+2}}\phi_{2i+2}(x)\right\} \quad P_m^{(\alpha+\beta-\delta)}(x)(1-x)^{\alpha+\beta-\delta}(1+x)^\delta dx \tag{13}$$

Substituting this value of A_m in equation (2) and interchanging the order of integration and summation, we get

$$\sum_{m=0}^{\infty} \frac{(\alpha+\beta-\delta+1)_m\Gamma(\beta+\mu+m+1)(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(m+\alpha+\beta-\delta+1)\Gamma(m+\delta+1)(\beta+1)_m\Gamma(\alpha-\mu+m+1)}$$

$$\sum_{i=0}^{\frac{n-2}{2}}\left\{\int_{a_{2i}}^{a_{2i+1}} f_{2i+1}(r) + \int_{a_{2i+1}}^{a_{2i+2}}\phi_{2i+2}(r)\right\} \cdot P_m^{(\alpha+\beta-\delta)}(x) \, P_m^{(\alpha,\beta)}(x)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta dr = f_j(x) \tag{14}$$
$$a_{j-1} < x < a_j \quad , j = 2,4,6,\ldots,n$$

$$\sum_{i=0}^{\frac{n-2}{2}}\int_{a_{2i+1}}^{a_{2i+2}} \phi_{2i+2}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta$$
$$\cdot S(r,x)dr = M_j(x) \quad a_{j-1} < x < a_j, \quad j = 2,4,6,\ldots,n.$$

where

$$M_j(x) = f_j(x) - \sum_{i=0}^{\frac{n-2}{2}}\int_{a_{2i}}^{a_{2i+1}} f_{2i+1}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta \cdot S(r,x)dr \tag{15}$$

for all $j = 2,4,6,\ldots,n.$

Taking $j = k$ in equation (14), where k is an even integer and $2 \le k \le n$ and n is the total number of considered equations, we get

$$\sum_{i=0}^{\frac{k-4}{2}}\left[\int_{a_{2i+1}}^{a_{2i+2}}\phi_{2i+2}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta(1-r)^{-\delta}S_r(r,x)dr+\int_{a_{k-1}}^{x}\phi_k(r)\cdot(1-r)^{\alpha+\beta-\delta}(1+r)^\delta(1-r)^{-\delta}S_r(r,x)dr\right.$$

$$\left.+\int_{x}^{a_k}\phi_k(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta\cdot(1-r)^{-\delta}S_r(r,x)dr\right] +$$

$$\sum_{i=\frac{k}{2}}^{\frac{n-2}{2}}\int_{a_{2i+1}}^{a_{2i+2}}\phi_{2i+2}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta(1-r)^{-\delta}S_r(r,x)dr = \frac{(1+x)^\beta}{a_n^{*}}M_k(x)$$

Or

$$\int_{\alpha_{k-1}}^{x} \emptyset_k(x)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta} \int_0^r \eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1} dy dr$$

$$+\int_x^{\alpha_k} \emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta}\int_0^x \eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1} dy dr$$

$$= \frac{(1+x)^\beta}{\alpha_n *} M_k(x) - \sum_{i=0}^{\frac{k-4}{2}}\int_{\alpha_{2i+1}}^{\alpha_{2i+2}} \emptyset_{2i+2}(r).(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta$$

$$\int_0^r \eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1}dydr - \sum_{i=k/2}^{\frac{n-2}{2}}\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\emptyset_{2i+2}(r).$$

$$(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\int_0^x \eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1}dydr.$$

Inverting the order of integration,

$$\int_{\alpha_{k-1}}^x \frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}}\int_y^{\alpha_k} \frac{\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} = \frac{(1+x)^\beta}{\alpha_n *}M_k(x) -$$

$$\int_0^{\alpha_{k-1}} \frac{\eta(y)\,dy}{(x-y)^{1-\beta-\mu+\delta}}\int_{\alpha_{k-1}}^{\alpha_k}\frac{\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}}$$

$$-\sum_{i=0}^{\frac{k-4}{2}}\left\{\int_0^{\alpha_{2i+1}}\frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}}\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}}\right.$$

$$\left.+\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}}\int_y^{\alpha_{2i+2}}\frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}}\right\} -$$

$$\sum_{i=k/2}^{\frac{n-2}{2}}\int_0^x\frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}}\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} \qquad (16)$$

Assuming

$$\int_y^{\alpha_k}\frac{\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} = \overline{\emptyset}_k(y), \alpha_{k-1}<x<\alpha_k \qquad (17)$$

For all $k = 2,4,6,\ldots,n$

with the help of the equation (17), the equation (16) takes the form

$$\int_{\alpha_{k-1}}^x\frac{\eta(y)\overline{\emptyset}_k(y)dy}{(x-y)^{1-\beta-\mu+\delta}} = \frac{(1+x)^\beta}{\alpha_n *}M_k(x) - \int_0^{\alpha_{k-1}}\frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}}$$

$$.\int_{\alpha_{k-1}}^{\alpha_k}\frac{\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} - \sum_{i=0}^{(k-4)/2}\left\{\int_0^{\alpha_{2i+1}}\frac{\eta(y)\,dy}{(x-y)^{1-\beta-\mu+\delta}}\right.$$

$$.\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} + \left.\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\frac{\overline{\emptyset}_{2i+2}(y)\eta(y)\,dy}{(x-y)^{1-\beta-\mu+\delta}}\right\}$$

$$-\sum_{i=k/2}^{(n-2)/2}\int_0^x\frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}}.\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} \qquad (18)$$

This equation is an Abel type integralequation and its solution is given by

$$\eta(t)\overline{\emptyset}_k(y) = F_k(y) - \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi\alpha_n *}\left[\int_0^{\alpha_{k-1}}\eta(\xi)d\xi\frac{d}{dy}\int_{\alpha_{k-1}}^y\frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right.$$

$$\int_{\alpha_{k-1}}^{\alpha_k}\frac{\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}} + \sum_{i=0}^{(k-4)/2}\left\{\int_0^{\alpha_{2i+1}}\eta(\xi)d\xi\frac{d}{dy}\int_{\alpha_{k-1}}^y\frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right.$$

$$\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}} + \left.\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\eta(\xi)\overline{\emptyset}_{2i+2}(\xi)d\xi.\frac{d}{dy}\int_{\alpha_{k-1}}^y\frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right\}$$

$$+\sum_{i=k/2}^{(n-2)/2}\frac{d}{dy}\int_{\alpha_{k-1}}^y\frac{dx}{(y-x)^{\beta+\mu-\delta}}\int_0^x\frac{\eta(\xi)\,d\xi}{(x-\xi)^{1-\beta-\mu+\delta}}$$

$$\left.\int_{\alpha_{2i+1}}^{\alpha_{2i+2}}\frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}}\right] \qquad (19)$$

where

$$F_k(y) = \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi \, a_n^*} \frac{d}{dy} \int_{a_{k-1}}^{y} \frac{(1+x)^\beta M_k(x)\,dx}{(y-x)^{\beta+\mu-\delta}} \qquad (20)$$

Changing the order of integration of the last integral of equation (26), we get

$$\eta(y)\bar{\emptyset}_k(y) = F_k(y) - \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi}\left[\int_0^{a_{k-1}} \eta(\xi)d\xi \frac{d}{dy} \int_{a_{k-1}}^{y} \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right.$$

$$\cdot \int_{a_{k-1}}^{a_k} \frac{\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}}$$

$$+ \sum_{i=0}^{(k-4)/2}\left\{\int_0^{a_{2i+1}} \eta(\xi)d\xi \frac{d}{dy}\cdot \int_{a_{k-1}}^{y} \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}} \int_{a_{2i+1}}^{a_{2i+2}} \frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}}\right.$$

$$+ \int_{a_{2i+1}}^{a_{2i+2}} \eta(\xi)\bar{\emptyset}_{2i+2}(\xi)d\xi \frac{d}{dy}\cdot \int_{a_{k-1}}^{y} \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right\} + \sum_{i=k/2}^{(n-2)/2}\left\{\int_0^{a_{k-1}} \eta(\xi)d\xi \frac{d}{dy}.\int_{a_{k-1}}^{y} \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right.$$

$$\cdot \int_{a_{2i+1}}^{a_{2i+2}} \frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}} + \cdot\frac{d}{dy} \int_{a_{k-1}}^{y} \eta(\xi)d\xi \int_{\xi}^{y} \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}$$

$$\cdot \int_{a_{2i+1}}^{a_{2i+2}}\frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}}\right\}\right] \qquad (21)$$

Using these equations,

$$\int_{\xi}^{y} \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}} = \frac{\pi}{\sin(1-\beta-\mu+\delta)\pi} \qquad (22)$$

And $\quad \dfrac{d}{dy}\displaystyle\int_a^y \dfrac{dx}{(y-x)^{1-m}(x-t)^m} = \dfrac{(a-t)^{1-m}}{(y-t)(y-a)^{1-m}} \qquad (23)$

Equation (21) will be ,

$$\eta(y)\bar{\emptyset}_k(y) = F_k(y) - \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi}\left[\int_0^{a_{k-1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}\,d\xi}{(y-a_{k-1})^{\beta+\mu-\delta}(y-\xi)}\right.$$

$$\cdot \int_{a_{k-1}}^{a_k} \frac{\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}} + \sum_{i=0}^{(k-4)/2}\left\{\int_0^{a_{2i+1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}\,d\xi}{(y-a_{k-1})^{\beta+\mu-\delta}(y-\xi)}\right.$$

$$\cdot \int_{a_{2i+1}}^{a_{2i+2}} \frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}} \int_{a_{2i+1}}^{a_{2i+2}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}\bar{\emptyset}_{2i+2}(\xi)\,d\xi}{(y-a_{k-1})^{\beta+\mu-\delta}(y-\xi)}\right\}$$

$$+ \sum_{i=k/2}^{(n-2)/2}\left[\int_0^{a_{k-1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}\,d\xi}{(y-a_{k-1})^{\beta+\mu-\delta}(y-\xi)} \int_{a_{2i+1}}^{a_{2i+2}} \frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}}\right.$$

$$- \sum_{i=k/2}^{(n-2)/2} \frac{d}{dy}\int_{a_{k-1}}^{y} \eta(\xi)d\xi \int_{a_{2i+1}}^{a_{2i+2}} \frac{\emptyset_{2i+2}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}} \qquad (24)$$

Equation (24) is also Abel type integral equation. Therefore its solution is given by

$$\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta = \frac{-\sin(1-\mu)\pi}{\pi}\frac{d}{dr}\int_y^{a_k} \frac{\bar{\emptyset}_k(y)\,dy}{(y-r)^\mu} \qquad (25)$$

for all $k = 2,4,6,\ldots,n.$

Therefore,

$$\int_{a_{k-1}}^{a_k} \frac{\emptyset_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-\xi)^{1-\mu}} = \frac{\sin(1-\mu)\pi}{\pi(a_{k-1}-\xi)^{-\mu}}\cdot\int_{a_{k-1}}^{a_k} \frac{\bar{\emptyset}_k(y)\,dy}{(y-\xi)(y-a_{k-1})^\mu} \qquad (26)$$

for all $k = 2,4,6,\ldots,n.$

Applying the above result in equation (24) and also applying the Leibnitz theorem we get

$$\eta(y)\bar{\emptyset}_k(y) = F_k(y) - \frac{\sin(1-\beta-\mu+\delta)\pi\sin(1-\mu)\pi}{\pi^2(y-a_{k-1})^{\beta+\mu-\delta}}\left[\int_0^{a_{k-1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(a_{k-1}-\xi)^{-\mu}} \int_{a_{k-1}}^{a_k} \frac{\bar{\emptyset}_k(t)\,dt}{(t-a_{k-1})^\mu(t-\xi)}\right.$$

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$$\sum_{i=0}^{(k-4)/2} \left\{ \int_0^{a_{2i+1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(a_{k-1}-\xi)^{-\mu}} \cdot \int_{a_{2i+1}}^{a_{2i+2}} \frac{\overline{\emptyset}_{2i+2}(t)\,dt}{(t-\xi)(t-a_{2i+1})^{\mu}} \right.$$

$$-\mu \int_{a_{2i+1}}^{a_{2i+2}} \overline{\emptyset}_{2i+2}(t)\,dt \int_{a_{2i+1}}^{t} \frac{dr}{(t-r)^{1+\mu}} \cdot \int_{a_{2i+1}}^{r} \left. \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(r-\xi)^{1-\mu}} \right\}$$

$$+ \sum_{i=k/2}^{\frac{n-2}{2}} \int_0^{a_{k-1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(a_{k-1}-\xi)^{-\mu}} \cdot \left. \int_{a_{2i+1}}^{a_{2i+2}} \frac{\overline{\emptyset}_{2i+2}(t)\,dt}{(t-\xi)(t-a_{2i+1})^{\mu}} \right]$$

$$- \sum_{i=k/2}^{(n-2)/2} \frac{\sin(1-\mu)\pi}{\pi} \cdot \frac{d}{dy} \int_{a_{k-1}}^{y} \eta(\xi)(a_{2i+1}-\xi)^{\mu}\,d\xi$$

$$\int_{a_{2i+1}}^{a_{2i+2}} \frac{\overline{\emptyset}_{2i+2}(t)\,dt}{(t-\xi)(t-a_{2i+1})^{\mu}} \tag{27}$$

This equation can be written as,

$$\eta(y)\overline{\emptyset}_k(y) = F_k(y) - \int_{a_{k-1}}^{a_k} \overline{\emptyset}_k(t)L_k(t,y)\,dt$$

$$- \sum_{i=0}^{(k-4)/2} \int_{a_{2i+1}}^{a_{2i+2}} \overline{\emptyset}_{2i+2}(t)R_{2i+2}(t,y)\,dt$$

$$- \sum_{i=k/2}^{(n-2)/2} \int_{a_{2i+1}}^{a_{2i+2}} \overline{\emptyset}_{2i+2}(t)T_{2i+2}(t,y)\,dt, \qquad a_{k-1} < r < a_k \tag{28}$$

where,

$$L_k(t,y) = \frac{\sin(1-\beta-\mu+\delta)\pi\sin(1-\mu)\pi}{\pi^2(y-a_{k-1})^{\beta+\mu-\delta}} \cdot \frac{1}{(t-a_{k-1})^{\mu}} \cdot \int_0^{a_{k-1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+2\mu-\delta}d\xi}{(y-\xi)(t-\xi)} \tag{29}$$

$$R_{2i+2}(t,y) = \frac{\sin(1-\beta-\mu+\delta)\pi\sin(1-\mu)\pi}{\pi^2(y-a_{k-1})^{\beta+\mu-\delta}} \left\{ \frac{1}{(t-a_{2i-1})^{\mu}} \cdot \int_0^{a_{2i+1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+2\mu-\delta}(a_{2i+1}-\xi)^{\mu}d\xi}{(y-\xi)(t-\xi)} \right.$$

$$\cdot \left. \int_{a_{2i+1}}^{r} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(r-\xi)^{1-\mu}} \right\} \tag{30}$$

$$T_{2i+2}(t,y) = \frac{\sin(1-\mu)\pi}{\pi} \frac{1}{(t-a_{2i+1})^{\mu}} \left\{ \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi(y-a_{k-1})^{\beta+\mu-\delta}} \int_0^{a_{k-1}} \frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}(a_{2i+1}-\xi)^{\mu}d\xi}{(y-\xi)(t-\xi)} \right.$$

$$\left. + \frac{d}{dy} \cdot \int_{a_{k-1}}^{y} \frac{\eta(\xi)(a_{2i+1}-\xi)^{\mu}d\xi}{(t-\xi)} \right\} \tag{31}$$

Substituting $k = 2,4,6,\ldots,n$ in equation (28) we will get n/2 simultaneous Fredholm Integral equations of the second kind with the help of these n/2 simultaneous equations we can calculate $\overline{\emptyset}_2(y), \overline{\emptyset}_4(y), \ldots \ldots \overline{\emptyset}_n(y)$ and Then the values of $\emptyset_2(x), \emptyset_4(x), \ldots \emptyset_n(x)$ can be determined. After all these calculations we can compute the coefficient A_m with the help of equation (13).

(ii) The solution of N-series equations of the second kind involving Jacobi polynomials of different indices :

Let us assume

$$\sum_{m=0}^{\infty} \frac{D_m P_m^{(\alpha,\beta)}(x)}{(\beta+1)_m \Gamma(\alpha-\mu+m+1)} = \Psi_i(x), \quad a_{i-1} < x < a_i \tag{32}$$

where, $i = 1,3,5,\ldots,n-1$. and where $\Psi_i(x)$ are unspecified functions. Using orthogonality relation it follows from equations (1) and (32)

$$D_m = \frac{(\alpha+\beta-\delta+1)_m \ \Gamma(\beta+\mu+m+1)(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(m+\alpha+\beta-\delta+1)\Gamma(m+\delta+1)}$$

$$\sum_{i=0}^{\frac{n-2}{2}} \left\{ \int_{a_{2i+1}}^{a_{2i+2}} f_{2i+2}(x) + \int_{a_{2i}}^{a_{2i+1}} \varphi_{2i+1}(x) \right\} \quad P_m^{(\alpha+\beta-\delta)}(x)(1-x)^{\alpha+\beta-\delta}(1+x)^{\beta}dx. \tag{33}$$

Substituting this value of D_m in equation (5) and interchanging the order of integration and summation, we get

$$\sum_{m=0}^{\infty} \frac{(\alpha+\beta-\delta+1)_m \Gamma(\beta+\mu+m+1)(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(m+\alpha+\beta-\delta+1)\Gamma(m+\delta+1)(\beta+1)_m\Gamma(\alpha-\mu+m+1)}$$

$$\sum_{i=0}^{\frac{n-2}{2}} \left\{ \int_{a_{2i}}^{a_{2i+1}} f_{2i+1}(r) + \int_{a_{2i+1}}^{a_{2i+2}} \Psi_{2i+2}(r) \right\} . P_m^{(\alpha+\beta-\delta)}(x) \ P_m^{(\alpha,\beta)}(x)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta dr = g_j(x)$$
(34)

$$a_{j-1} < x < a_j \ , \ j = 1,3,5,\ldots,n-1$$

$$\sum_{i=0}^{\frac{n-2}{2}} \int_{a_{2i}}^{a_{2i+1}} \Psi_{2i+1}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta . S(r,x) dr = N_j(x)$$

$$a_{j-1} < x < a_{j,} \ \ j = 1,3,5,\ldots,n-1.$$
(35)

where

$$N_j(x) = g_j(x) - \sum_{i=0}^{\frac{n-2}{2}} \int_{a_{2i+1}}^{a_{2i+2}} f_{2i+2}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta S(r,x) dr$$
(36)

for all $j = 1,3,5,\ldots,n-1.$

Taking $j = k$ in equation (34), where k is an odd integer and $1 \leq k \leq n-1$ and n is the total number of considered equations, we get

$$\sum_{i=0}^{\frac{k-3}{2}} \left[\int_{a_{2i}}^{a_{2i+1}} \Psi_{2i+1}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta .(1-r)^{-\delta}S_r(r,x) dr + \int_{a_{k-1}}^{x} \Psi_k(r).(1-r)^{\alpha+\beta-\delta}(1+r)^\delta (1-r)^{-\delta}S_r(r,x) dr \right.$$
$$\left. + \int_x^{a_k} \Psi_k(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta .(1-r)^{-\delta}S_r(r,x) dr \right] +$$

$$\sum_{i=\frac{k+1}{2}}^{\frac{n-2}{2}} \int_{a_{2i}}^{a_{2i+1}} \Psi_{2i+1}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta .(1-r)^{-\delta}S_r(r,x) dr = \frac{(1+x)^\beta}{a_n *} N_k(x)$$

Or

$$\int_{a_{k-1}}^{x} \Psi_k(x)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta \int_0^r \eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1} dy dr$$
$$+ \int_x^{a_k} \Psi_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta \int_0^x \eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1} dy dr$$

$$= \frac{(1+x)^\beta}{a_n *} N_k(x) - \sum_{i=0}^{\frac{k-3}{2}} \int_{a_{2i}}^{a_{2i+1}} \Psi_{2i+1}(r).(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta$$

$$\int_0^r \eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1} dy dr -$$

$$\sum_{i=(k+1)/2}^{\frac{n-2}{2}} \int_{a_{2i}}^{a_{2i+1}} \Psi_{2i+1}(r).(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta$$
$$\int_0^x \eta(y)(r-y)^{\mu-1}(x-y)^{\beta+\mu-\delta-1} dy dr.$$

Inverting the order of integration,

$$\int_{a_{k-1}}^{x} \frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}} \int_y^{a_k} \frac{\Psi_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} = \frac{(1+x)^\beta}{a_k *} N_k(x) -$$
$$\int_0^{a_{k-1}} \frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}} \int_{a_{k-1}}^{a_k} \frac{\Psi_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} - \sum_{i=0}^{\frac{k-3}{2}} \left\{ \int_0^{a_{2i}} \frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}} . \right.$$
$$\int_{a_{2i}}^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} + \int_{a_{2i}}^{a_{2i+1}} \frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}} \int_y^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} \right\} - \sum_{i=(k+1)/2}^{\frac{n-2}{2}} \int_0^x \frac{\eta(y)dy}{(x-y)^{1-\beta-\mu+\delta}}$$
$$\int_{a_{2i}}^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}}$$
(37)

Assuming

$$\int_y^{a_k} \frac{\Psi_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta dr}{(r-y)^{1-\mu}} = \bar{\Psi}_k(y), a_{k-1} < x < a_k$$
(38)

For all $k = 1,3,5,\ldots,n-1.$
With the help of the equation (38), the equation (37) takes the form,

$$\int_{a_{k-1}}^{x} \frac{\eta(y)\overline{\Psi}_k(y)\,dy}{(x-y)^{1-\beta-\mu+\delta}} = \frac{(1+x)^\beta}{a_n*}N_k(x) - \int_0^{a_{k-1}} \frac{\eta(y)\,dy}{(x-y)^{1-\beta-\mu+\delta}}$$

$$\cdot \int_{a_{k-1}}^{a_k} \frac{\Psi_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\,dr}{(r-y)^{1-\mu}} - \sum_{i=1}^{(k-3)/2}\left\{\int_0^{a_{2i}} \frac{\eta(y)\,dy}{(x-y)^{1-\beta-\mu+\delta}}\right.$$

$$\left. \cdot \int_{a_{2i}}^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\,dr}{(r-y)^{1-\mu}} + \int_{a_{2i}}^{a_{2i+1}} \frac{\overline{\Psi}_{2i+1}(y)\eta(y)\,dy}{(x-y)^{1-\beta-\mu+\delta}}\right\}$$

$$-\sum_{i=(k+1)/2}^{(n-2)/2}\int_0^x \frac{\eta(y)\,dy}{(x-y)^{1-\beta-\mu+\delta}}\cdot\int_{a_{2i}}^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-\delta}(1+r)^\delta\,dr}{(r-y)^{1-\mu}} \qquad (39)$$

This equation is an Abel type integral equation and its solution is given by

$$\eta(t)\overline{\Psi}_k(y) = G_k(y) - \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi a_n*}\left[\int_0^{a_{k-1}} \eta(\xi)\,d\xi \frac{d}{dy}\right.$$

$$\int_{a_{k-1}}^y \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}} \int_{a_{k-1}}^{a_k} \frac{\Psi_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\,dr}{(r-\xi)^{1-\mu}} +$$

$$\sum_{i=1}^{(k-3)/2}\left\{\int_0^{a_{2i}} \eta(\xi)\,d\xi \frac{d}{dy} \int_{a_{k-1}}^y \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right.$$

$$\int_{a_{2i}}^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\,dr}{(r-\xi)^{1-\mu}} + \int_{a_{2i}}^{a_{2i+1}} \eta(\xi)\overline{\Psi}_{2i+1}(\xi)\,d\xi$$

$$\left.\cdot\frac{d}{dy}\int_{a_{k-1}}^y \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right\} + \sum_{i=(k+1)/2}^{(n-2)/2}\frac{d}{dy}\int_{a_{k-1}}^y \frac{dx}{(y-x)^{\beta+\mu-\delta}}$$

$$\left.\int_0^x \frac{\eta(\xi)\,d\xi}{(x-\xi)^{1-\beta-\mu+\delta}}\cdot\int_{a_{2i}}^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\,dr}{(r-\xi)^{1-\mu}}\right] \qquad (40)$$

where

$$G_k(y) = \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi a_n^*}\frac{d}{dy}\int_{a_{k-1}}^y \frac{(1+x)^\beta N_k(x)\,dx}{(y-x)^{\beta+\mu-\delta}} \qquad (41)$$

Changing the order of integration of the last integral of equation (40), we get

$$\eta(y)\overline{\Psi}_k(y) = G_k(y) - \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi}\left[\int_0^{a_{k-1}} \eta(\xi)\,d\xi \frac{d}{dy}\right.$$

$$\int_{a_{k-1}}^y \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\cdot\int_{a_{k-1}}^{a_k} \frac{\Psi_k(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\,dr}{(r-\xi)^{1-\mu}}$$

$$+\sum_{i=1}^{(k-3)/2}\left\{\int_0^{a_{2i}} \eta(\xi)\,d\xi \frac{d}{dy}\int_{a_{k-1}}^y \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right.$$

$$\int_{a_{2i}}^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\,dr}{(r-\xi)^{1-\mu}} + \int_{a_{2i}}^{a_{2i+1}} \eta(\xi)\overline{\Psi}_{2i+1}(\xi)\,d\xi$$

$$\frac{d}{dy}\cdot\int_{a_{k-1}}^y \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\right\} + \sum_{i=(k+1)/2}^{(n-2)/2}\left\{\int_0^{a_{k-1}} \eta(\xi)\,d\xi \frac{d}{dy}\cdot\right.$$

$$\int_{a_{k-1}}^y \frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\cdot\int_{a_{2i}}^{a_{2i+1}} \frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^\delta\,dr}{(r-\xi)^{1-\mu}} +$$

$$\frac{d}{dy}\int_{a_{k-1}}^{y}\eta(\xi)d\xi\int_{\xi}^{y}\frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}\cdot\int_{a_{2i}}^{a_{2i+1}}\frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta}dr}{(r-\xi)^{1-\mu}}\Big\}\Big]$$

(42)

Using these equations,

$$\int_{\xi}^{y}\frac{dx}{(y-x)^{\beta+\mu-\delta}(x-\xi)^{1-\beta-\mu+\delta}}=\frac{\pi}{\sin(1-\beta-\mu+\delta)\pi}$$

(43)

And

$$\frac{d}{dy}\int_{a}^{y}\frac{dx}{(y-x)^{1-m}(x-t)^{m}}=\frac{(a-t)^{1-m}}{(y-t)(y-a)^{1-m}}$$

(44)

Equation (42) will be ,

$$\eta(y)\overline{\Psi}_{k}(y)=G_{k}(y)-\frac{\sin(1-\beta-\mu+\delta)\pi}{\pi}\Bigg[\int_{0}^{a_{k-1}}\frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}\,d\xi}{(y-a_{k-1})^{\beta+\mu-\delta}(y-\xi)}$$
$$\cdot\int_{a_{k-1}}^{a_{k}}\frac{\Psi_{k}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta}dr}{(r-\xi)^{1-\mu}}+\sum_{i=1}^{(k-3)/2}\Bigg\{\int_{0}^{a_{2i}}\frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}\,d\xi}{(y-a_{k-1})^{\beta+\mu-\delta}(y-\xi)}$$
$$\int_{a_{2i}}^{a_{2i+1}}\frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta}dr}{(r-\xi)^{1-\mu}}\int_{a_{2i}}^{a_{2i+1}}\frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}\overline{\Psi}_{2i+1}(\xi)\,d\xi}{(y-a_{k-1})^{\beta+\mu-\delta}(y-\xi)}\Bigg\}$$
$$+\sum_{i=(k+1)/2}^{(n-2)/2}\Bigg[\int_{0}^{a_{k-1}}\frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}\,d\xi}{(y-a_{k-1})^{\beta+\mu-\delta}(y-\xi)}\cdot\int_{a_{2i}}^{a_{2i+1}}\frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta}dr}{(r-\xi)^{1-\mu}}\Bigg]$$
$$-\sum_{i=(k+1)/2}^{(n-2)/2}\frac{d}{dy}\int_{a_{k-1}}^{y}\eta(\xi)d\xi\int_{a_{2i}}^{a_{2i+1}}\frac{\Psi_{2i+1}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta}dr}{(r-\xi)^{1-\mu}}$$

(45)

Equation (45) is also Abel type integral equation. Therefore its solution is given by

$$\Psi_{k}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta}=\frac{-\sin(1-\mu)\pi}{\pi}\frac{d}{dr}\int_{r}^{a_{k}}\frac{\Psi_{k}(y)\,dy}{(y-r)^{\mu}}$$

(46)

for all $k=1,3,5,\ldots,n-1.$

Therefore,

$$\int_{a_{k-1}}^{a_{k}}\frac{\Psi_{k}(r)(1-r)^{\alpha+\beta-2\delta}(1+r)^{\delta}dr}{(r-\xi)^{1-\mu}}=\frac{\sin(1-\mu)\pi}{\pi(a_{k-1}-\xi)^{-\mu}}\cdot\int_{a_{k-1}}^{a_{k}}\frac{\Psi_{k}(y)\,dy}{(y-\xi)(y-a_{k-1})^{\mu}}$$

(47)

for all $k=1,3,5,\ldots,n-1.$

Applying the above result in equation (45) and also applying the Leibnitz theorem, we get

$$\eta(y)\overline{\Psi}_{k}(y)=G_{k}(y)-\frac{\sin(1-\beta-\mu+\delta)\pi\sin(1-\mu)\pi}{\pi^{2}(y-a_{k-1})^{\beta+\mu-\delta}}\Bigg[\int_{0}^{a_{k-1}}\frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(a_{k-1}-\xi)^{-\mu}}$$
$$\int_{a_{k-1}}^{a_{k}}\frac{\overline{\Psi}_{k}(t)\,dt}{(t-a_{k-1})^{\mu}(t-\xi)}\sum_{i=1}^{(k-3)/2}\Bigg\{\int_{0}^{a_{2i}}\frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(a_{2i}-\xi)^{-\mu}}\int_{a_{2i}}^{a_{2i+1}}\frac{\overline{\Psi}_{2i+1}(t)\,dt}{(t-\xi)(t-a_{2i})^{\mu}}$$
$$-\mu\int_{a_{2i}}^{a_{2i+1}}\overline{\Psi}_{2i+1}(t)\,dt\int_{a_{2i}}^{t}\frac{dr}{(t-r)^{1+\mu}}\int_{a_{2i}}^{r}\frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(r-\xi)^{1-\mu}}\Bigg\}$$
$$+\sum_{i=(k+1)/2}^{\frac{n-2}{2}}\int_{0}^{a_{k-1}}\frac{\eta(\xi)(a_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(a_{2i}-\xi)^{-\mu}}\int_{a_{2i}}^{a_{2i+1}}\frac{\overline{\Psi}_{2i+1}(t)\,dt}{(t-\xi)(t-a_{2i})^{\mu}}\Bigg]-$$
$$\sum_{i=(k+1)/2}^{(n-2)/2}\frac{\sin(1-\mu)\pi}{\pi}\cdot\frac{d}{dy}\int_{a_{k-1}}^{y}\eta(\xi)(a_{2i}-\xi)^{\mu}\int_{a_{2i}}^{a_{2i+1}}\frac{\overline{\Psi}_{2i+1}(t)\,dt}{(t-\xi)(t-a_{2i})^{\mu}}$$

(48)

This equation can be written as,

$$\eta(y)\overline{\Psi}_k(y) = G_k(y) - \int_{\alpha_{k-1}}^{\alpha_k} \overline{\Psi}_k(t)P_k(t,y)\,dt$$

$$- \sum_{i=1}^{(k-3)/2} \int_{\alpha_{2i}}^{\alpha_{2i+1}} \overline{\Psi}_{2i+1}(t)Q_{2i+1}(t,y)\,dt - \sum_{i=(k+1)/2}^{(n-2)/2} \int_{\alpha_{2i}}^{\alpha_{2i+1}} \overline{\Psi}_{2i+1}(t)S_{2i+1}(t,y)\,dt,$$

$$\alpha_{k-1} < r < \alpha_k \tag{49}$$

Where,

$$P_k(t,y) = \frac{\sin(1-\beta-\mu+\delta)\pi\sin(1-\mu)\pi}{\pi^2(y-\alpha_{k-1})^{\beta+\mu-\delta}} \cdot \frac{1}{(t-\alpha_{k-1})^{\mu}} \int_0^{\alpha_{k-1}} \frac{\eta(\xi)(\alpha_{k-1}-\xi)^{\beta+2\mu-\delta}d\xi}{(y-\xi)(t-\xi)} \tag{50}$$

$$Q_{2i+1}(t,y) = \frac{\sin(1-\beta-\mu+\delta)\pi\sin(1-\mu)\pi}{\pi^2(y-\alpha_{k-1})^{\beta+\mu-\delta}} \left\{ \frac{1}{(t-\alpha_{2i-1})^{\mu}} \int_0^{\alpha_{2i}} \frac{\eta(\xi)(\alpha_{k-1}-\xi)^{\beta+2\mu-\delta}(\alpha_{2i}-\xi)^{\mu}d\xi}{(y-\xi)(t-\xi)} \right.$$

$$\left. -\mu \int_{\alpha_{2i}}^t \frac{dr}{(t-r)^{1+\mu}} \int_{\alpha_{2i}}^r \frac{\eta(\xi)(\alpha_{k-1}-\xi)^{\beta+\mu-\delta}d\xi}{(y-\xi)(r-\xi)^{1-\mu}} \right\} \tag{51}$$

$$S_{2i+1}(t,y) = \frac{\sin(1-\mu)\pi}{\pi} \frac{1}{(t-\alpha_{2i})^{\mu}} \left\{ \frac{\sin(1-\beta-\mu+\delta)\pi}{\pi(y-\alpha_{k-1})^{\beta+\mu-\delta}} \int_0^{\alpha_{k-1}} \frac{\eta(\xi)(\alpha_{k-1}-\xi)^{\beta+\mu-\delta}(\alpha_{2i}-\xi)^{\mu}d\xi}{(y-\xi)(t-\xi)} \right.$$

$$\left. + \frac{d}{dy} \cdot \int_{\alpha_{k-1}}^y \frac{\eta(\xi)(\alpha_{2i}-\xi)^{\mu}d\xi}{(t-\xi)} \right\} \tag{52}$$

Substituting $k = 1,3,5,\ldots,n-1$ in equation (49) we will get n/2 simultaneous Fredholm Integral equations of the second kind. With the help of these n/2 simultaneous equations we can calculate $\overline{\Psi}_1(y), \overline{\Psi}_3(y), \ldots \ldots \overline{\Psi}_{n-1}(y)$ and then the values of $\Psi_1(t), \Psi_3(t), \ldots \Psi_{n-1}(t)$ can be determined .After all these calculations we can compute the coefficient D_m with the help of equation (33).

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