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RINGS WITH ASSOCIATORS IN THE COMMUTATIVE CENTER

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Abstract: Thedy has introduced the subject of rings which satisfy the identity $[(R, R, R), R] = 0$ and which satisfy one additional identity such as $(x, x, x) = 0$. By assuming char. $\neq 2, 3$ and simplicity, Thedy proved that R must be either commutative or associative. Kleinfeld proved that the additional identity assumed by Thedy is not necessary. Using Kleinfeld's method, Suvarna et.al prove that if R is a simple ring of char. $\neq 3$ satisfying $[(R, R, R), R] = 0$, then $(x, x, x) = 0$ for all x in R. From this R is either commutative or associative. This paper gives an alter proof of suvarna's method.

Keywords: Simple ring, center, char. $\neq n$., nucleus

INTRODUCTION

Throughout this paper R represents a nonassociative ring satisfying the identity$[(R, R, R), R] = 0$.
(1)
A ring R is simple if A is an ideal of R, then either $A = 0$ or $A = R$. A ring is of char. $\neq n$ if $nx = 0$ implies $x = 0$ for every x in R and n a natural number. The nucleus $N(R)$ of a ring R is the set of all elements n in R such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. The center U of R is defined as $U = \{u \in R/[u, R] = 0\}$. From (1) it follows that all associators are in the center U.
In every arbitrary ring the following identities are satisfied:
$[xy, z] + [yz, x] + [zx, y] = (x, y, z) + (y, z, x) + (z, x, y)$, (2)
$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$, (3)
and
$[xy, z] = x[y, z] + [x, z]y + (x, y, z) + (z, x, y) - (x, z, y)$. (4)First we prove the following properties of R.

Lemma 1 **:** If R satisfies $[(R, R, R), R] = 0$ and $V = \{v \in U/vR \subset U\}$, then V is an ideal of R such that $(x, y, v) \in V$ and $(v, y, x) \in V$ for $v \in U$ and all x, y in R.

Proof **:** Since $V \subset U$ it is sufficient to show V is a right ideal. Let $v \in V$. Then for all $r, s \in R$, $vr \in U$ follows from the definition of V. Since (1) implies $(v, r, s) \in U$ and $(vr)s = (v, r, s) + v(rs) \in U$, it follows that $vr \in V$. Thus V is a right ideal and hence it is an ideal of R.
From (3) and (1), we get
$z(x, y, v) = (zx, y, v) - (z, xy, v) + (z, x, yv) - (z, x, y)v \in U$.
Similarly we get
$z(v, y, x) = (v, y, x)z \in U$.
Hence $(x, y, v) \in V$ and $(v, y, x) \in V$.

Lemma 2 **:** The canonical homomorphism of R onto R/V maps U into the center of R/V.

Proof **:** Let $x, y \in R$ and $v \in U$. We know that $[x, v] = 0$, $(x, y, v) \in V$ and $(v, y, x) \in V$ from Lemma 1. Therefore from (4), we get

$$(x, v, y) = (x, y, v) + (v, x, y) - [xy, v] + [x, v]y + x[y, v]$$
$$= (x, y, v) + (v, x, y) \in V.$$

Lemma 3 **:** $(x, y, z)^3 \equiv (x, x, x)\,(y, y, y)\,(z, z, z) \bmod V$.

Proof **:** Since U is mapped into the center of R/V, we have modulo V, that

$$(x, y, z)^3 \equiv (x, y, z)\,(x(x, y, z), y, z)$$
$$\equiv -(x, y, z)\,((x, x, y)z, y, z)$$
$$\equiv -(x(x, x, y), y, z)\,(z, y, z)$$
$$\equiv (x, x, x)\,(y, y, z)\,(z, y, z).$$

Now
$$(y, y, z)\,(z, y, z) \equiv (z(y, y, z), y, z) \equiv -((z, y, y)z, y, z)$$
$$\equiv -(z, (z, y, y)y, z) \equiv (z, z(y, y, y), z)$$
$$\equiv (y, y, y)\,(z, z, z).$$

Therefore $(x, y, z)^3 = (x, x, x)\,(y, y, y)\,(z, z, z) \bmod V$.

Now we prove the additional identity $(x, x, x) = 0$ assumed by Thedy.

Theorem 1 **:** If R is a simple ring of char. $\neq 3$ satisfying $[(R, R, R), R] = 0$, then $(x, x, x) = 0$ for all x in R.

Proof **:** We assume that R is not commutative. Hence $V \neq R$. Since R is simple, then because of the Lemma 1, we are reduced to the case $V = 0$. By commuting each term in (3) with r and using (1), we obtain

$$[w(x, y, z), r] = -[(w, x, y)z, r] = -[z(w, x, y), r].$$

By permuting $(wyzx)$ cyclically, we obtain
$[w(x, y, z), r] = -[z(w, x, y), r] = [y(z, w, x), r] = -[x(y, z, w), r]$. (5)
By substituting $y = x$ and $z = a$ in (4), where a is an arbitrary associator and using (1), we get

$$(x, x, a) + (a, x, x) - (x, a, x) = 0.$$

Now multiplying the terms with x on left and commuting with z, we obtain

$$[x(x, x, a) + x(a, x, x) - x(x, a, x), z] = 0. \quad (6)$$

Using (5) in (6), we have

$$-[a(x, x, x), z] - [a(x, x, x), z] - [a(x, x, x), z] = 0,$$

that is, $-3[a(x, x, x), z] = 0$. Since R is of char. $\neq 3$, this implies

$$[a(x, x, x), z] = 0.$$

Now we replace a with (b, c, d). Then we have
$[(b, c, d)(x, x, x), z] = 0$. Using (1), we can write it as
$$[(x, x, x)(b, c, d), z] = 0. \quad (7)$$
By applying (5) to (7), we obtain
$[b(c, d, (x, x, x)), z] = 0 = [c(d, (x, x, x), b), z] = [d((x, x, x), b, c), z]$.
This and (1) prove that $(c, d, (x, x, x)) \in V$, $(d, (x, x, x), b) \in V$ and

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$((x, x, x), b, c) \in V$. Since $V = 0$, (x, x, x) must be in the nucleus $N(R)$ of R. Now we substitute $x = r$, $y = s$ and $z = (x, x, x)$ in (2). Using $(x, x, x) \in N(R)$ and (1), we obtain

$$[(x, x, x)r, s)] = -[s(x, x, x), r] = [rs, (x, x, x)] = 0.$$

So $(x, x, x)r \in U$. That is, $(x, x, x) \in V$. Since $V = 0$, it follows that $(x, x, x) = 0$.

Now we prove Thedy's result without additional condition.

***Theorem 2*:** Let R be a simple ring of char. $\neq 3$ satisfying $[(R, R, R), R] = 0$. Then R is either commutative or associative.

***Proof* :** The ideal V of Lemma 1 is contained in the center U of R. Since R is simple either $V = R$ or $V = 0$. In the first case R is commutative. Next we consider the case $V = 0$. From Theorem 1 and Lemma 3, we have $(x, y, z)^3 = 0$. Thus the associators are in the center and are nilpotent. Therefore $R(x, y, z)$ is a nilpotent ideal of R. Hence $R(x, y, z) = 0$. This implies that $(x, y, z) \in V$. Since $V = 0$, it follows that $(x, y, z) = 0$. Hence R is associative.

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