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ON THE REDUCED INTERSECTION GRAPH OF A RING Z_n

Shaik Sajana
Department of Mathematics,
S.V. University,
Tirupati, A.P., India-517502.

D. Bharathi
Department of Mathematics,
S.V. University,
Tirupati, A.P., India-517502.

K.K. Srimitra
Department of Mathematics,
S.V. University,
Tirupati, A.P., India-517502.

Abstract: For the ring integers modulo Z_n, we define the Reduced Intersection Graph $G^*(Z_n)$, is a simple undirected graph, whose vertex set is the nonzero ideals of Z_n and two distinct ideals I and J are adjacent if and only if they have nonzero intersection, i.e., $I \cap J \neq \emptyset$. In this paper, we investigate the connectedness of the graph $G^*(Z_n)$. Also, we compute the radius, diameter, girth and also domination number of the reduced intersection graph. Further, we compare these parameters for the existed Intersection graph $G^{'}(Z_n)$ and the reduced intersection graph $G^*(Z_n)$.

Keywords: Intersection graph; Reduced intersection graph; Radius; Diameter; Girth; Domination number; Ideals

I. INTRODUCTION

Recently, many mathematicians have been investigating the relationship between ring-theoretic aspects and graph-theoretic aspects. The algebraic structure of the ring using graph-theoretic properties associated to them has become an interesting topic in the last few years. For such kind of study, researchers defined graphs whose vertices are set of elements of a ring [4], set of nonzero elements of a ring [1], set of nonzero zero-divisors [2, 3, 11], set of ideals of a ring [8], etc and the edges are defined with respect to a condition on the elements of the vertex set.

The Intersection graph was first introduced by Bosak [6] arising from semigroups, he considered sub semigroups of a semigroup S are vertices and in which two distinct vertices are adjacent if and only if they have nontrivial intersection, and proved that S is a non denumerable semigroup or a periodic semigroup with more than two elements, then its graph $G(S)$ is connected. Various constructions of the intersection graphs related to different structures are found in [9, 7] and [10, 11].

In [5], the intersection graph $G^{'}(Z_n)$ for the ring of integers modulo n is defined as a simple undirected graph, whose vertices are the nonzero elements of Z_n and in which two distinct vertices are adjacent if and only if their corresponding principal ideals having nonzero intersection in Z_n.

The following figure.1 illustrates the intersection graph of the ring Z_4.

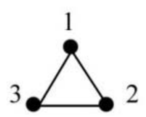

Fig. 1. The graph $G^{'}(Z_4)$

II. DEFINITIONS AND NOTATIONS

For the graph G, the two distinct vertices x and y are adjacent, we write $x - y$. The graph G is called connected, if there exist at least one path between every pair of distinct vertices in G. Also G is called complete, if every vertex is adjacent to remaining all the vertices in G. The distance between two vertices x, y is the length of the shortest path between these vertices, denoted by $d(x, y)$. The eccentricity $ecc(v)$ of a vertex v in the graph G is the distance between v and a vertex farthest from v in G. Radius of the graph G is the smallest eccentricity among the vertices of G, is denoted by $rad(G)$. Diameter of a graph G is the greatest eccentricity among all the vertices of G, is denoted by $diam(G)$. The length of a shortest cycle in the graph G is called girth of G, denoted by $gr(G)$. A subset D of the vertex set $V(G)$ is said to be a dominating set of G, if every vertex in $V(G) - D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$.

In the ring Z_n, the sets of regular elements, zero-divisors and nonzero zero-divisors are denoted by $reg(Z_n)$, $Z(Z_n)$ and $Z(Z_n)^*$ respectively. Also, the trivial ideals are (0) and $Z_n = (u)$, for all units u in Z_n. The canonical representation of a positive integer $n > 1$ can be written as $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes and $\alpha_i \geq 1$, $\forall\ 1 \leq i \leq k$, $k \geq 1$. If a/m and b/m, then m is called common multiple of a and b. And m is called least common multiple of a and b, if m is a common multiple of a and b and also m_0 is any other common multiple of a and b , then m/m_0, we write $m = lcm(a, b)$, for all integers a, b, m, m_0.

III. REDUCED INTERSECTION GRAPH

From the definition of the intersection graph of a finite ring Z_n, any two distinct vertices that generate the same ideal will play similar roles within the structure of the graph and this explains in the following Theorem 1.

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Theorem 1. If $(x) = (y)$, for some $x, y \in {Z_n}^*$, then x is adjacent to z in $G^{'}(Z_n)$ if and only if y is adjacent to z in $G^{'}(Z_n)$, for all $z \in {Z_n}^*$.

Proof. Obviously $x - y$, for $(x) = (y)$. Suppose that $x - z$, for some $z \in {Z_n}^* \Leftrightarrow (x) \cap (z) \neq \{0\}$

$\Leftrightarrow (y) \cap (z) \neq \{0\}$, since $(x) = (y)$

$\Leftrightarrow y - z$

This shows that any two elements that generate the same ideal will have exactly the same set of neighbours. In this way, $G^{'}(Z_n)$ is somewhat redundant in its portrayal of relationships between the principal ideals. The inclusion of multiple generators of the same ideal presents only to complicate the graph larger. From this, the reduced intersection graph $G^*(Z_n)$ of the ring Z_n is more efficient analog to the graph $G^{'}(Z_n)$ and is defined as the set of nonzero principal ideals $\Omega(Z_n)^*$ of the ring Z_n as vertices and two distinct ideals I and J are adjacent if and only if $I \cap J \neq \{0\}$.

The following figure 2 and 3 shows the intersection graph and the reduced intersection graph of the ring Z_8 respectively.

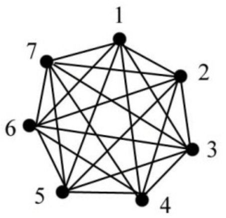

Fig. 2. The graph $G^{'}(Z_8)$

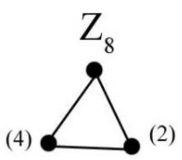

Fig. 3. The graph $G^*(Z_8)$

There are more advantages to studying the reduced intersection graphs $G^*(Z_n)$ as compared to the intersection graphs $G^{'}(Z_n)$. From the above figure the graph $G^*(Z_8)$ is much simpler than the graph $G^{'}(Z_8)$, which provides the same information regarding principal ideal relationships.

Lemma 2. In the graph $G^*(Z_n)$, the vertex Z_n is adjacent to the remaining all the vertices.

Proof. The proof follows trivially, since $I \cap Z_n \neq \{0\}$, for every vertex $I \neq Z_n \in V(G^*(Z_n))$.

Theorem 3. For each $n > 1$, then the graph $G^*(Z_n)$ is connected.

Proof. The proof directely follows from the above Lemma 2.

Corollary 4. For $n = p$, p is prime, then the graph $G^*(Z_n)$ is trivial.

Theorem 5. For $n = p^m$, p is prime and $m > 1$, then the graph $G^*(Z_n)$ is complete.

Proof. Let I and J be two distinct arbitrary vertices in the graph $G^*(Z_n)$.

If I is adjacent to J, then there is nothing to prove.

If I is not adjacent to J, then I and J can be written as $I = (p^r)$ and $J = (p^s)$, where $r \neq s$ and $1 \leq r, s < m$.

This implies that $I \cap J = (p^t) \neq \{0\}$, where $t = \max\{r, s\}$. This completes the proof.

Theorem 6. For every $n \neq p$, p is prime, then the radius and diameter of the graph $G^*(Z_n)$ is

$$rad\big(G^*(Z_n)\big) = 1 \text{ and } diam\big(G^*(Z_n)\big) = \begin{cases} 1, & if\ n = p^m \\ 2, & if\ n \neq p^m \end{cases}.$$

Proof.

Case 1. If $n = p^m$, then the graph $G^*(Z_n)$ is complete from the above theorem 5.

This implies that $ecc(I) = 1$, $\forall\, I \in V(G^*(Z_n))$. Hence $rad\big(G^*(Z_n)\big) = diam\big(G^*(Z_n)\big) = 1$.

Case 2. Assume that $n \neq p^m$.

From the above Lemma 2, we have $ecc(Z_n) = 1$.

Let I and J be any two distinct arbitrary vertices in the graph $G^*(Z_n)$.

i. If I is adjacent to J, then we have $d(I, J) = 1$.

ii. If I is not adjacent to J, then from the above Lemma 2, we have there exists a vertex Z_n in $G^*(Z_n)$ such that $I - Z_n - J$. Thus, we have $d(I, J)$ and hence $ecc(I) = 2$.

Thus, we have $rad\big(G^*(Z_n)\big) = 1$ and $diam\big(G^*(Z_n)\big) = 2$.

Theorem 7. For $n \neq p$, p is prime, then $rad\big(G^*(Z_n)\big) = rad\Big(G^{'}(Z_n)\Big)$ and $diam\big(G^*(Z_n)\big) = diam\Big(G^{'}(Z_n)\Big)$.

Proof. The proof follows from above Theorem 6 and the proof of Theorem 5 in [5].

Remark. Except $p = 2$, we have $G^{'}\big(Z_p\big)$ is complete with at least 2 vertices and hence $rad\Big(G^{'}(Z_n)\Big) = diam\Big(G^{'}(Z_n)\Big) = 1$. But $G^*\big(Z_p\big)$ is trivial, for every prime p and also $G^{'}(Z_n)$ is trivial for $n = p$.

Theorem 8. For $n \neq p$, p is prime, then the girth of the graph $G^*(Z_n)$ is

$$gr\big(G^*(Z_n)\big) = \begin{cases} \infty, & if\ n = p^2,\ pq \\ 3, & otherwise \end{cases}.$$

Proof.

Case 1. If $n = p^2$, pq, then obviously from the following figures 4 and 5, we have $gr\big(G^*(Z_n)\big) = \infty$.

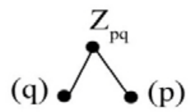

Fig. 4. The graph $G^*(Z_{p^2})$

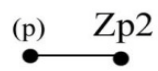

Fig. 5. The graph $G^*(Z_{pq})$

Case 2. If $n \neq p^2, pq$, then we have

1. For $n = p^m$, $m > 2$, then there exists a cycle $Z_n - (p) - (p^2) - Z_n$ of length 3 in $G^*(Z_n)$.
2. For $n = p_1 p_2 \ldots p_m$, $m > 2$ and
3. For $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$, where $p_1 < p_2 < \cdots < p_m$ are primes, α_i is a positive integer and $\sum \alpha_i > m$, $\forall\, 1 \leq i \leq m$ and $m > 1$, then there exists a cycle $Z_n - (p_1) - (p_1 p_2) - Z_n$ of length 3 in $G^*(Z_n)$.

This shows that $gr\big(G^*(Z_n)\big) = 3$.

Theorem 9. For $n \neq p$, p^2, pq, then $gr\big(G^*(Z_n)\big) = gr\Big(G^{'}(Z_n)\Big)$.

Proof. The proof follows from above Theorem 8 and the proof of Theorem 6 in [5].

Remark.

i. Except for $p = 2, 3$, we have the graph $G^{'}\big(Z_p\big)$ is complete with $(p-1) > 3$ elements, thus $gr\Big(G^{'}\big(Z_p\big)\Big) = 3$. But the graph $G^*\big(Z_p\big)$ is trivial and also $gr\Big(G^{'}\big(Z_p\big)\Big) = \infty$, since $G^{'}\big(Z_p\big)$ is K_1 and K_2 respectively for $p = 2, 3$.

ii. If $n = p^2$ and pq, then the graph $G^{'}(Z_n)$ contains $p^2 - 1$ and $pq - 1$ vertices respectively with at least 2 unit elements, and thus $gr\Big(G^{'}(Z_n)\Big) = 3$. But the

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graphs $G^*(Z_{p^2})$ and $G^*(Z_{pq})$ are K_2 and $K_{1,2}$ respectively.

Theorem 10. The domination number of the graph $G^*(Z_n)$ is $\gamma\big(G^*(Z_n)\big) = 1$.

Proof. Obviously follows from the above Lemma 2.

Examples.

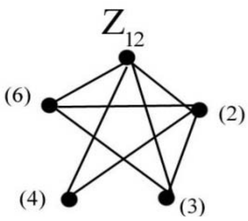

Fig. 6. The graph $G^*(Z_{12})$

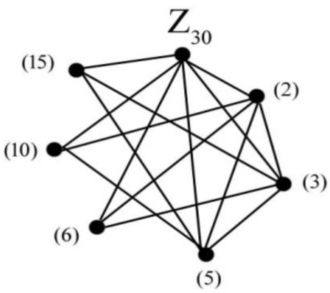

Fig. 7. The graph $G^*(Z_{30})$

IV. REFERENCES

[1] M. Afkhami and K. Khashyarmanesh, “The Cozero-divisor Graph of a Commutative Ring”, Southeast Asian Bulletin of Mathematics 35 (2011), 753-762.

[2] S. Akbari, H.R. Maimani and S. Yassemi, “When a zero-divisor graph is planar or a complete r-partite graph”, J. Algebra 270 (2003) 169–180.

[3] D. F. Anderson and Philip S. Livingston, “The Zero-Divisor Graph of a Commutative Ring”, J. of Algebra 217(1999), 434-447.

[4] I. Beck, “Coloring of commutative rings”, J. of Algebra 116(1988), 208-226.

[5] D. Bharathi, Shaik Sajana, “Some properties of the intersection graph for finite Commutative Rings”, International Journal of Scientific and Innovative Mathematical Research, Volume 3, Special Issue 3, July 2015, PP 1062-1066.

[6] Bohdan Zelinka, Liberec, “Intersecion Graphs of Finite Abelian Groups”, Czechoslovak Mathematical journal, V.25, No 2(1975), 171-174.

[7] J. Bosak, “The graphs of semigroups, in Theory of graphs and its applications”, Proc. Sympos. Smolenice (June1963), Academic Press, New York, 1965, 119-125.

[8] Ivy Chakrabarty, Shamik Ghosh, T.K. Mukherjee, M.K. Sen, “Intersection Graphs of Ideals of Rings”, Electronic Notes in Discrete Mathematics, 23(2005), 23-32.

[9] Ivy Chakrabarty, Shamik Ghosh, T.K. Mukherjee, M.K. Sen, “Intersection Graphs of Ideals of Rings”, Discrete Mathematics, 309(2009), 5381-5392.

[10] B. Csakany and G. Pollak, “The Graph of Subgroups of a Finite Group”, Czechoslovak Math. J., 19(1969), 241-247.

[11] Shaik Sajana, K.K. Srimitra, D. Bharathi, “Intersection Graph of Zero-divisors of a Finite Commutative Ring”, International Journal of Pure and Applied Mathematics, Volume 109, No. 7(2016), 51-58.

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