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TOTAL ZERO DIVISOR GRAPHS OF POLYNOMIAL RINGS

D. Bharathi
Department of Mathematics,
S. V. University,
Tirupati.

D. Eswara Rao
Department of Mathematics,
S. V. University,
Tirupati.

Abstract: In this paper, we study the total zero divisor graph of polynomial rings. In this if Z(R)[x] is an ideal of R[x], then we discuss the completeness of Z(Γ(R[x])) and Z(Γ(R[[x]])) and also we find diam(Z(Γ(R[x]))) = 3. Further we prove that let R be a finite commutative ring such that Z(R) is not an ideal of R then Reg(Γ(R[x]) is connected and diam(Reg(Γ(R[x])) ≤ 2.

Key words: Total zero divisor graph, Polynomial ring, Commutative ring, Diameter, Regular graph.

I. INTRODUCTION

Michael Axtell, J.Coykendall [3], studied zero divisor graph of polynomials and power series over commutative rings. They examine the preservation of the zero divisor graphs under extensions to polynomial rings and power series rings, and also discussed the girth and completeness of polynomial rings and power series rings. R[x] and R[[x]] represent standard notation for the polynomial ring and formal power series ring. It is known that Z(R) ⊆ Z(R[x]) ⊆ Z(R)[x] always hold. The second inclusion may be proper, for example 2 + 3x ∈ Z(Z_6)[x]\ Z(Z_6[x]). It is clear that the first inclusion may be proper as well. The well known McCoy's theorem gives the description of the set of zero divisors in a polynomial rings. We havef(x) ∈ Z(R[x]) if and only if there exist r ∈ R* such that rf(x) = 0. Therefore not only the coefficients have to be zero divisors, but the ideal generated by these coefficients should be a non zero annihilator.

Let us first suppose that Z(R[x]) is an ideal of R[x]. It is evident that in this case the sub graph of zero divisors Z[Γ(R[x])] is complete. One has Z(R[x]) = Z(R)[x] consequently.

R[x] /Z(R[x]) = R[x] /Z(R)[x] ≅ (R/Z(R))[x]

The right hand side is a polynomial ring which clearly cannot be isomorphic to Z_2 or to Z_3 and we conclude that Reg(Γ(R[x])) is not connected by [2].

Akbari and Heydari [1], studied the Regular graph of a non commutative ring. He proved that if R is a Reduced left Neotherian ring and 2 ∉ Z(R), then the chromatic number and clique number of Reg(Γ(R)) are the same and they are 2^r, where r is the number of minimal prime ideal of R.

Asir and Tamizh chelvam [6] studied some properties of the total graphs and its complement of commutative rings.

For Ring theoretic concepts we refer to Kaplansky [5] and Graph theoretic terminology we refer to Bondy and Murthy [4].

II. MAIN THEOREMS

Lemma 1: Let R be a commutative ring and let f(x) = $\sum_{i=0}^{\infty} f_i\, x^i$ ∈ R[[x]] if for some natural t , f_t is regular in R while f_i is nilpotent for 0 ≤ i ≤ t-1 then f(x) is regular in R[[x]]. ♦

As a corollary we have the following useful results. We have stated it only for the non trivial case that neither polynomial is 0.

Corollary 2: If f(x) and g(x) are nonzero zero-divisors of R[x]. Then the following are equivalent

(i) (f(x), g(x)) ⊆ Z(R[x]).

(ii) f(x) and g(x) have a common non zero annihilator in R[x].

(iii) There is a non zero element r∈ R such that rf(x) = 0= rg(x).

(iv) If deg(f(x)) = n, then f(x) + x^{n+1} g(x) is a zero divisor of R[x]. ♦

Theorem 3: If Z(R[x]) is an ideal of R[x]. Then the following are equivalent

(i) Z[Γ(R[[x]])] is complete.

(ii) Z[Γ(R[x])] is complete.

(iii) Z[Γ(R)] is complete.

Proof: Clearly (i) ⇒ (ii) ⇒ (iii)

For (iii) ⇒ (i) let us assume that Z[Γ(R)] is complete.

Since Z(R) is an ideal of R we have x+y∈ Z(R) for every x, y ∈ Z(R)*.

Let f(x), g(x) ∈ Z(R[[x]]) be distinct.

By lemma 1 no coefficient of f(x) or g(x) can be regular since all non regular coefficients of f and g are nilpotent.

Thus all coefficients of f(x) and g(x) are zero divisors in R.

Since f(x) ∈ Z(R[[x]]).

By McCoy theorem ∃ r ∈ R* such that r f(x) = 0.

Similarly s g(x) = 0 for s ∈ R*.

Let t = (s, r) and t/s ,t/r, ⇒ t(f(x)+g(x)) = tf(x) + t g(x) = 0.

∴ f(x)+g(x) ∈ Z(R[[x]]) so f(x) and g(x) are adjacent .

∴ Z[Γ(R[[x]])] is complete. ♦

Remark 4: If Z(R) is not an ideal of R then Z[Γ(R)] is complete and hence diam(Z[Γ (R[x])]) = 2 and diam(Z[Γ(R[[x]])]) = 2. ♦

Lemma 5: Let R be a ring such that Z(R) is not an ideal of R then ⟨ Z(R[x])⟩ = R[x] if and only if ⟨Z(R)⟩ = R.

Proof: Suppose that ⟨ Z(R[x])⟩ = R[x].

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Therefore, there exist polynomials $f_1(x), f_2(x), f_3(x),\ldots, f_n(x) \in Z(R[x])$ such that $f_1(x) + f_2(x) + f_3(x) +\ldots + f_n(x) = 1$.
It follows that $Z_1 + Z_2 + Z_3 +\ldots+ Z_n = 1$. Where $Z_1, Z_2, Z_3, \ldots, Z_n$ are constant coefficients of the previous polynomials, since $Z[R[x]] \subseteq Z(R)[x]$.
All coefficients of the polynomials are zero divisors therefore $Z_1, Z_2, Z_3, \ldots, Z_n \in Z(R)$ as well.
So $R = \langle Z(R)\rangle = \langle Z_1, Z_2, Z_3, \ldots, Z_n \rangle$
The other implication is trivial. ♦

The following theorem we show that diam(Reg(Γ(R[x]))

Theorem 6: Let R be a ring such that Z(R) is an ideal of R, if Reg(Γ(R[x]) is connected then diam(Reg(Γ(R[x])) ≤ 2.
Proof: Let Reg(Γ(R[x]) is connected
If it is sufficient to show that if f_1, f_2, $f_3$and f_4 are distinct vertices of Reg(Γ(R[x]) and there is a path f_1-f_2-f_3-f_4 form f_1 to f_2 then f_1 and f_2 are adjacent.
Now $f_1+f_2, f_2+f_3, f_3+f_4 \in Z(R[x])$
It implies that $f_1+f_4 = (f_1+f_2)-(f_2+f_3)-(f_3+f_4) \in Z(R[x])$ since Z(R(x)) is an ideal of R(x)
Thus f_1 and f_4 are adjacent.
Therefore diam(Reg(Γ(R[x])) ≤ 2. ♦

Remark 7: Let R be a finite commutative ring such that Z(R) is not an ideal of R and let f ∈ Z(R[x]) − Reg(R[x]) then the powers of x will constitute a semi group, and for each f ∈ Z(R[x]) − Reg(R[x]), there are similar results as above i.e. f ∈ Z(R[x]) − Reg(R[x]) is adjacent to t elements of Reg(R[x]) and s elements of Z(R[x]). Also each f ∈ Nil(R[x]) is adjacent to all elements of Z(R[x]), therefore elements of Reg(R[x]), Nil(R[x])) and semi group of powers of f in which f ∈ Z(R[x]) − Nil(R[x]) have similar properties. ♦

In the following theorem we show that diam(Reg(Γ(R[x]))) ≤ 2.

Theorem 8: Let R be a finite commutative ring such that Z(R) is not an ideal of R then Reg(Γ(R[x]) is connected and diam(Reg(Γ(R[x])) ≤ 2.
Proof: Let f ∈ Z(R[x]) − Nil(R[x]).
Then powers of f will constitute a semi group and we conclude that R has nontrivial idempotent element.
Since R is finite then according to lemma $R[x] \cong R_1[x]\times R_2[x]$.
Note that $Reg(R[x]) = Reg(R_1[x]\times R_2[x])$
$= Reg(R_1[x]) \times Reg(R_2[x])$

So for distinct (f, g), (p, q) ∈ $Reg(R_1[x]\times R_2[x])$, (f, g) − (-f, -g) − (p, q) is a path of length at most two in Reg(R[x]).
Thus Reg(R[x]) is connected with diam(Reg(Γ(R[x])) ≤ 2. ♦

Lemma 9: Let R be a commutative ring there are nontrivial rings R_1 and R_2 such that $R \simeq R_1\times R_2$ if and only if there exists a non trivial idempotent e ∈ R. In this case one can choose R_1 = Re and R_2 = R(1− e). ♦

According to the previous lemma if we consider a finite commutative ring containing idempotent elements, then we can write it as a product of two rings.
We use this lemma and we prove the following theorem.

Theorem 10: Let R be a commutative ring if R has a non trivial idempotent then T(Γ(R[x]) is connected with diam (TΓ(R[x])) = 2.
Proof: Let e(x) ∈ R(x)\{0, 1} be idempotent
Then R(x) = (e(x), 1− e(x)) with e(x), 1− e(x) Z(R(x)).
So we claim is clear by Theorems 3.3 and 3.4 in [2] respectively. ♦

Theorem 11: Let R be a commutative ring such that Z(R) is not an ideal of R then Z(Γ(R[x]) is connected with diam(T(Γ(R[x])) = 2.
Proof: For every f(x), g(x) ∈ Z(R[x])* is adjacent to 0 thus f – 0 – g is a path in Z(Γ(R[x]) of length 2 between any two distinct f, g ∈ Z(R[x])*.
Moreover there are non adjacent vertices f, g ∈ Z(R[x])*.
Since Z(R[x]) is not an ideal of R[x].So diam(T(Γ(R[x])) = 2. ♦

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