Volume 8, No. 6, July 2017 (Special Issue III)
International Journal of Advanced Research in Computer Science
RESEARCH PAPER
Available Online at www.ijarcs.info
ISSN No. 0976-5697

BALANCED LAPLACIAN ENERGY OF A FRIENDSHIP GRAPH

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Abstract : Let G be a signed connected graph with order n and size m. The signed laplacian $\bar{L}$ is defined by $\bar{L} = \bar{D} - W$, where $\bar{D}$ is signed degree matrix and W is a symmetric matrix with zero diagonal entries. The signed laplacian is a symmetric positive semidefinite. Let $\mu_1 \geq \mu_2 \geq \ldots \mu_{n-1} \geq \mu_n = 0$ be the eigen values of the laplacian matrix. The signed laplacian energy is defined as $\bar{L}E(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. In this paper, we defined balanced signed laplacian energy of a Friendship graph. We also attained their upper bounds.

Subject Classification: 05C50, 05C69

Key words: Balanced signed graph, Signed Laplacian matrix, Signed laplacian Energy of a Friendship graph.

I INTRODUCTION

A Signed graph is a graph with the additional structure that edges are given a sign of either +1 or -1. Formally, a signed graph is a pair $\sum = (\Gamma,\sigma)$ consisting of an underlying graph $\Gamma = (V,E)$ and a signature $\sigma \rightarrow \{ E \quad +1,-1\}$[9,14,18,19,20]. We define the adjacency matrix $A(\sum) = (a_{ij})_{nxn}$ as

$$a_{ij} = \begin{cases} \sigma(e_{ij}) \text{ if } v_i \text{ is adjacent to } v_j \\ 0 \text{ otherwise} \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$, of adjacency matrix A are assumed in non increasing order. The energy E(G) of G is defined to be the sum of the absolute values of the eigenvalues of G.

i.e., $E(G) = \sum_{i=1}^{n}|\lambda_i|$ [1,2,3,5,7].

I.Gutman and B.zhou [4,5,6,8,10,17] defined the Laplacian energy of a graph G in the year 2006. Let G be a finite, connected graph with order n vertices and size m respectively. The Laplacian matrix of the graph G, denoted by $L(G) = D(G) - A(G)$ is a square matrix of order n, where D(G) is the diagonal matrix of vertex degrees of the graph G and A(G) is the adjacency matrix. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian spectrum of its Laplacian matrix G[13,15,16] then the Laplacian energy LE(G) of G is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

A. Definitions and Examples

Definition 1.1

Given a signed graph G = (V, W) (where W is a symmetric matrix with zero diagonal entries), the underlying graph of G is the graph with the vertex set V and the set of (undirected) edges

$E = \{ (v_i, v_j) \; w_{ij} \neq 0\}$.

Definition 1.2

Let G be a graph with order n and size m. The **Laplacian matrix** of the graph G is denoted by $L = (L_{ij})$ is a square matrix of order n whose elements are defined as

$$L_{ij} = \begin{cases} -1 \text{ if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 \text{ if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i \text{ if } i = j \end{cases}$$

where d_i is the degree of the vertex v_i

Definition 1.3

Let (V,W) be a signed graph where W is a (mxm) symmetric matrix with zero diagonal entries and with the other entries $w_{ij} \in R$ arbitrary. The degree of any vertex v_i is defined

as $\bar{d}_i = \bar{d}(v_i) = \sum_{j=1}^{m}|w_{ij}|$ and the **signed degree matrix** $\bar{D} = \text{diag}(\bar{d}(v_1), \bar{d}(v_2), \ldots \bar{d}(v_m))$.

Definition 1.4

The **Signed Laplacian** $\bar{L}$ is defined by $\bar{L} = \bar{D} - W$, where $\bar{D}$ is signed degree matrix. The signed Laplacian is symmetric positive semi definite.

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Definition 1.5

Let G = (V, W) be a signed graph whose underlying graph is connected. Then G is **balanced** if there is a partition of its vertex set V into two clusters V_1 and V_2 such that all the positive edges connect vertices within V_1 or V_2 and all the negative edges connect vertices between V_1 and V_2. If the signed graph has even number of negative edges then it is called a **Balanced Signed graph.**

Definition 1.6

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $\bar{L}$, which are called Signed Laplacian eigenvalues of G. The **Signed Laplacian energy** $\bar{L}E(G)$ of G is defined as $\bar{L}E(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$, where $\frac{2m}{n}$ is the average degree of the graph G.

Example 1.7

Signed graph is an ordered pair (G, σ), where G = (V, E) is a graph with the vertex set V and the edge set E. Let $\sigma : E \rightarrow \{p, n\}$ is a sign function, the edges with the sign p are positive and the edges with the sign n are negative. Positive edges are drawn with bold lines and the negative edges are drawn with dotted lines.

In a signed graph, if it is possible to partition the vertex set V into two clusters such that every edge that connects two vertices that belong to the same cluster is positive and every edge that connects two vertices that belong to different clusters is negative then we call the signed graph **partitionable** (or) **clusterable.**

In fig 1.1, let the vertex set V be partitioned into two clusters ({1,2,4,7,8}, {3,5,6,9}) in which positive edges are represented in bold lines and negative edges are denoted by dotted lines.

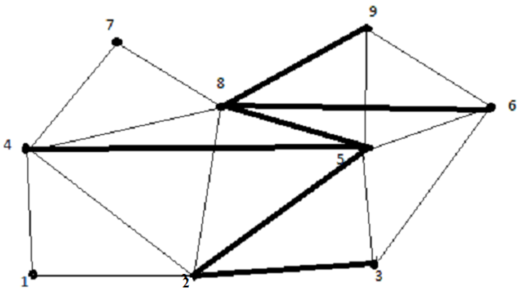

Fig 1.1 Graph G

Every connected balanced graph can be characterized as signed graph in which every cycle has an even number of negative edges. The balanced signed laplacian matrix of the graph G is given by

$$\bar{L}(G) = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 5 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 5 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 6 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & -1 & 6 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 3 \end{pmatrix}_{(9x9)}$$

The eigenvalues are 0, 1.4790, 1.7513, 2.7883, 4.3570, 4.8815, 6.2158, 7.2159, 7.3112. The signed laplacian energy of the graph is

$\bar{L}E(G) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|$, where the average degree of the graph is 3.56. Hence the laplacian energy of the graph G is 20.4028 approximately.

II BALANCED SIGNED LAPLACIAN ENERGY OF A FRIENDSHIP GRAPH

Definition 2.1

The Windmill graph Wd(k, n) is an undirected graph constructed for k≥ 2 and n≥ 2 by joining n copies of the complete graph k_k at a shared vertex. That is, it is a 1-clique sum of these complete graph. By construction, the windmill graph Wd(3,n) is the friendship graph F_n, the windmill graph Wd(2,n) is the star graph S_n and the windmill graph Wd(3,2) is the butterfly graph.

A. Laplacian Energy Of Friendship Graph

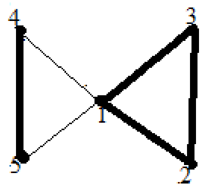

Fig 3.1

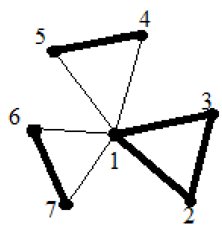

Fig 3.2

In Fig 3.1 and 3.2, the butterfly graph and the fan graph are partitioned into two clusters. The vertices 1,2 and 3 are in one of the clusters and the remaining vertices are in another cluster. The vertices which are connected within the clusters are given positive edges and the vertices which are connected between different clusters are given negative edges. The Balanced signed Laplacian matrix of the Butterfly graph Wd(3,2) is given by

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$$\bar{L}(Wd(3,2)) = \begin{pmatrix} 4 & -1 & -1 & 1 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The characteristic equation of the balanced signed Laplacian matrix is μ^5-12μ^4+50μ^3-89μ^2+45μ = 0 and the eigen values are 0,1,3,3 and 5.

The Balanced signed Laplacian matrix of Fan graph Wd(3,3) is given by

$$\bar{L}(Wd(3,3)) = \begin{pmatrix} 6 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix}$$

The characteristic equation of the Balanced signed Laplacian matrix is μ^7-18μ^6+123μ^5-412μ^4+711μ^3-594μ^2+189μ = 0 and the eigen values are 0,1,1,3,3,3 and 7.

The above result can be extended to finite number of cliques(n copies) with the shared vertex.

The Balanced signed Laplacian matrix of windmill graph Wd(3,n) is

$$\bar{L}(Wd(3,n)) = \begin{pmatrix} 2n & -1 & -1 & \cdots & 1 & 1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ -1 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & & & & & \\ \cdots & & & & & \\ 1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{(2n+1) X (2n+1)}$$

The characteristic equation of the Laplacian matrix $\bar{L}$(Wd(3,n)) is

$\mu(\mu-(2n+1))(\mu-1)^{n-1}((\mu-3)^n = 0.$

The balanced Laplacian energy of the Friendship graph $\bar{L}$E(Wd(3,n)) is

$= \left|0-\frac{6n}{2n+1}\right| + \left|(2n+1)-\frac{6n}{2n+1}\right| + \left|1-\frac{6n}{2n+1}\right|$(n-1) times + $\left|3-\frac{6n}{2n+1}\right|$n times

$= \frac{12n}{2n+1}.$

B. Observations

Some bounds of Laplacian energy that are needed for the following proof

1. Let G be a graph with order n and size m and p components (p ≥ 1) then LE(G) ≤ $\frac{2m}{n}$p + $\sqrt{(n-p)[2M-p(\frac{2m}{n})^2]}$. [4]

2. Let G be an undirected simple and connected graph with n, n≥2 vertices and m edges, then $\sum_{i=1}^{n-1}\mu_i = \sum_{i=1}^{n} d_i$ = 2m and $\sum_{i=1}^{n-1}\mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n-1} d_i$ = M_1 +2m where M_1 is the sum of squares of the vertex usually refered to as the first Zagreb index.[11]

3. Let G be an undirected simple and connected graph with n, n≥2 vertices and m edges, then $\frac{M_1}{m} \geq 2\sqrt{\frac{M_1}{n}} \geq \frac{4m}{n}$.[8]

4. Let G be an (n,m) graph such that n ≥3 and m ≥ 1 then
LE(G) ≥ $\mu_1 - \mu_{n-1} + \frac{2m}{n}$ with equality if and only if n=3 or for $\mu_2 = \mu_3 \ldots = \mu_{n-2} = \frac{2m}{n}$. [11]

III BOUNDS ON BALANCED LAPLACIAN ENERGY OF A FRIENDSHIP GRAPH

Let G be a Friendship graph with order 2n+1 and size 3n and it contains n components of complete graph K_3 then the upper bound on balanced laplacian energy of Friendship graph is

$\bar{L}$ E(G) ≤ $\frac{6n^2}{2n+1} + \sqrt{(n+1)[2M-\frac{36n^3}{(2n+1)^2}]}$

where M = m + $\frac{1}{2}\sum_{i=1}^{n}(d_i-\frac{2m}{n})^2$.

Proof:

Let G be a (n,m) graph contains p components (p≥ 1) and the Laplacian eigenvalues are arranged non increasing order $\mu_1 \geq \mu_2 \geq, \ldots \geq \mu_n$ then the upper bound laplacian energy of the graph G is

LE(G) ≤ $\frac{2m}{n}$p + $\sqrt{(n-p)[2M-p(\frac{2m}{n})^2]}$ by result 1 and the Friendship graph has order 2n+1 and size 3n and the number of components are n.

$$\bar{L} E(G) \leq \frac{6n.n}{2n+1} + \sqrt{(2n+1-n)[2M-n(\frac{6n}{2n+1})^2]}$$

$$\bar{L} E(G) \leq \frac{6n^2}{2n+1} + \sqrt{(n+1)[2M-\frac{36n^3}{(2n+1)^2}]}.$$

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