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On Intuitionistic Fuzzy Rough Sets

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Abstract: Rough set theory, fuzzy set theory and Intuitionistic fuzzy set theory deal with sets having imprecise boundaries. The contemporary concern about unravelling hidden knowledge from imperfect data and incomplete information systems has lead to many hybrid theories which combine rough set theory with fuzzy set theory and intuitionistic fuzzy set theory. In this paper, a new type of intuitionistic fuzzy rough set is introduced by extending the definition of fuzzy rough sets proposed by A. Nakamura, into the intuitionistic fuzzy context. The properties of these approximations are explored. A decomposition theorem for the proposed intuitionistic fuzzy rough lower and upper approximations is presented. Further, it is proved that they coincide with Pawlak's rough set approximations in the crisp case.

Keywords: Approximations; Rough Set; Fuzzy Approximation Space; Intuitionistic Fuzzy Set; Intuitionistic Fuzzy Rough Set.

I. INTRODUCTION

The rough set theory, proposed by Pawlak [20] in 1982 is one of the effective mathematical tools to handle imperfect knowledge. Since its inception, there have been intensive studies, both in application and theoretical point of view. The rough set approach has been successfully applied in artificial intelligence and cognitive sciences, particularly in machine learning, knowledge based systems, decision analysis, data mining, feature selection, image processing and pattern recognition. Rough set theoretical tools are used to find efficient algorithms to extract hidden patterns from data, determine minimal sets of data and generate decision rules from information systems.

Rough set theory is often compared with Fuzzy set theory proposed by L. A. Zadeh [27], as both of them deal with sets having imprecise boundaries. Many researchers combined the two concepts and defined fuzzy rough set and rough fuzzy set. The first attempt to define fuzzy rough set in a fuzzy approximation space was made by A. Nakamura in 1988 [18]. D. Dubois and H. Prade [11] gave another definition by incorporating the membership values of the fuzzy equivalence relation in the definition of fuzzy rough approximations. Subsequently, extensive research has been done in this direction and many extensions of Dubois and Prade's definition have been proposed [1, 7, 10, 11, 13, 17, 22, 25, 26]. D. Boixader, J. Jacas, and J. Recasens [5] studied the approximations of a fuzzy subset with respect to an indistinguishability operator. Fuzzification of Iwinski's concept of rough sets was done by S. Nanda, and S. Majumdar [19]. Fuzzy rough set theory has found applications in many fields such as feature selection, web content categorization, systems monitoring, expert systems and decision analysis.

The intuitionistic fuzzy set (IFS) theory launched by K. T. Atanassov [2, 3], addresses the problem of uncertainty by considering a non-membership function along with the fuzzy membership function on a universal set. IFS theory got access to a wide area of knowledge where fuzzy set concepts cannot be applied. Many authors studied the combination of rough set theory and IFS theory [6, 9, 12, 15, 23, 24, 25, 28, 29, 30, 31]. However, generalization of fuzzy rough sets proposed in [18] has not been done yet.

In this paper, the concept of α-intuitionistic fuzzy rough sets is introduced by extending the definition of of fuzzy rough sets given by A. Nakamura [18], into the IFS context. The properties of α-intuitionistic fuzzy rough approximations are studied in detail. It is also verified that the α-intuitionistic fuzzy rough approximations coincide with Pawlak's rough set approximations in the crisp case. A decomposition theorem for the proposed approximations in terms of the α-cuts of the fuzzy equivalence relation and the (a, b)-cuts of the IFS is also presented. Further it is proved that every fuzzy equivalence relation R on U induces an intuitionistic fuzzy topology on U in which the α-intuitionistic fuzzy rough lower and upper approximations act as the interior and closure operators respectively. It also determines three complete distributive lattices of IFSs in U.

This paper is organised as follows: Section 2 reviews the background work on intuitionistic fuzzy rough set theory, section 3 deals with the proposed definition and general properties of the α-intuitionistic fuzzy rough approximations, section 4 describes the set theoretic operations, section 5 provides the topological and lattice theoretic properties and section 5 concludes the paper.

II. RELATED WORK

A brief review of the different definitions of intuitionistic fuzzy rough sets existing in the literature is presented in this section. For the basic notions of fuzzy set theory, rough set theor and IFS theory, the readers may refer to [16], [21] and [4, 8] respectively.

A. Fuzzy Rough Sets

Fuzzy rough sets encapsulate the related but distinct concepts of vagueness and indiscernibility. A fuzzy rough set consists of a pair of fuzzy membership functions which correspond to the fuzzy lower and upper approximations of a fuzzy set in a fuzzy approximation space. Only the definitions proposed by the A. Nakamura[18] is presented here.

A *fuzzy approximation space* is a pair (U, R), where U is a non-empty set of objects and R is a fuzzy equivalence relation. The *fuzzy rough lower and upper approximations* of a fuzzy set A on U [18] are defined as

$$\mu_{\underline{R^\alpha}(A)}(x) = \bigwedge_{R(x,y)\geq\alpha} \mu_A(y) \quad (1)$$
$$\mu_{\overline{R}^\alpha(A)}(x) = \bigvee_{R(x,y)\geq\alpha} \mu_A(y) \quad (2)$$

respectively. Here, $\wedge$ and $\vee$ represent infimum and supremum respectively.

B. Intuitionistic Fuzzy Rough Sets

Merging of rough set theory with IFS theory was initiated by K. Chakrabarty, T. Gedeon, and L. Koczy [6]. Following Iwinski's point of view of a rough set [14], they constructed an intuitionistic fuzzy rough set (A, B) of a rough set (P, Q), where A and B are both IFSs on U such that $A \subseteq B$. Later, this definition was renamed as rough intuitionistic fuzzy sets by S. K. Samanta, and T. K. Mondal [23] and they defined intuitionistic fuzzy rough sets as a couple (A, B) where both A and B are fuzzy rough sets in the sense of Nanda and Majumdar [19] and $A \subseteq B^C$. S. P. Jena, and S. K. Ghosh [15] have also worked on intuitionistic fuzzy rough sets in the context of Iwinski's rough sets. C. Cornelis, M. D. Cock, and E. E. Kerre [9] generalised the definition of fuzzy rough sets proposed by A. Radzikowska, and E. E. Kerre [22] into intuitionistic fuzzy rough sets. This was further generalized by L. Zhou, W. Z. Wu, and W. X. Zhang [30] by replacing the intuitionistic fuzzy equivalence relation by an intuitionistic fuzzy binary relation. They presented an axiomatic characterization for abstract intuitionistic fuzzy approximation operators to be a lower and an upper approximation operator. B. K. Tripathy [25] defined rough sets on the approximation space generated by the (α, β)-cuts of an intuitionistic fuzzy proximity relation R. L. Zhou, and W. Z. Wu [28] introduced intuitionistic rough fuzzy sets as the approximations of an IFS with respect to a crisp reflexive relation and intuitionistic fuzzy rough sets as the approximations of an IFS with respect to a fuzzy reflexive relation. The concept of fuzzy rough sets proposed by Dubois and Prade [11] was generalised by L. Zhou, W. Z. Wu, and W. X. Zhang [29] by considering an intuitionistic fuzzy binary relation. They also studied the topological properties of intuitionistic fuzzy rough sets. The relationship between intuitionistic fuzzy rough approximation operators based on an intuitionistic fuzzy reflexive and transitive relation and IF topological spaces were explored by T. Feng, , S. P. Zhang, J. S. Mi, and Y. Li [12].

The proposed definition of α-intuitionistic fuzzy rough set approximation and their properties are given in the following section.

III. α-INTUITIONISTIC FUZZY ROUGH SETS

Let (U, R) be a fuzzy approximation space, where U is a non-empty set of objects and R is a fuzzy equivalence relation on U. Let $A = \{(x,\ \mu_A(x),\ \vartheta_A(x)) : x \in U\}$ be an IFS in U.

Definition 3.1:

For each $\alpha \in (0,1]$, the *α-intuitionistic fuzzy rough lower and upper approximations* of A are defined respectively as

$$\underline{R^\alpha}(A) = \{(x, \mu_{\underline{R^\alpha}(A)}(x), \vartheta_{\underline{R^\alpha}(A)}(x)) : x \in U\} \text{ and} \quad (3)$$

$$\overline{R}^\alpha(A) = \{(x, \mu_{\overline{R}^\alpha(A)}(x), \vartheta_{\overline{R}^\alpha(A)}(x)) : x \in U\}, \quad (4)$$

where $\mu_{\underline{R^\alpha}(A)}$, $\vartheta_{\underline{R^\alpha}(A)}(x)$, $\mu_{\overline{R}^\alpha(A)}$ and $\vartheta_{\overline{R}^\alpha(A)}$ are functions from U to $[0,1]$, given by

$$\mu_{\underline{R^\alpha}(A)}(x) = \bigwedge_{R(x,y)\geq\alpha} \mu_A(y) \quad (5)$$
$$\vartheta_{\underline{R^\alpha}(A)}(x) = \bigvee_{R(x,y)\geq\alpha} \vartheta_A(y) \quad (6)$$
$$\mu_{\overline{R}^\alpha(A)}(x) = \bigvee_{R(x,y)\geq\alpha} \mu_A(y) \quad (7)$$
$$\vartheta_{\overline{R}^\alpha(A)}(x) = \bigwedge_{R(x,y)\geq\alpha} \vartheta_A(y) \quad (8)$$

The IFS $\overline{R}^\alpha(A) - \underline{R^\alpha}(A)$ corresponds to the boundary region of A. An IFS A on U is called an *α-intuitionistic fuzzy rough set* iff $\underline{R^\alpha}(A) \neq \overline{R}^\alpha(A)$. Here, $\wedge$ and $\vee$ represent infimum and supremum respectively.

The following proposition shows that the above defined approximations are IFSs on U.

Proposition 3.1:

Let (U, R) be a fuzzy approximation space. For $A \in IFS(U)$, $\underline{R^\alpha}(A)$ and $\overline{R}^\alpha(A)$ are IFSs in U.

Proof:

Since A is an IFS on U, $\mu_A(y) \in [0,1]$, $\vartheta_A(y) \in [0,1]$ and $\mu_A(y) + \vartheta_A(y) \leq 1, \forall y \in U$. Hence $\mu_{\underline{R^\alpha}(A)}(y)$, $\mu_{\overline{R}^\alpha(A)}(y)$, $\vartheta_{\underline{R^\alpha}(A)}(y)$ and $\vartheta_{\overline{R}^\alpha(A)}(y)$ are elements of $[0, 1]$, $\forall y \in U$. Let $x \in U$. Then,

$$\mu_A(y) + \vartheta_A(y) \leq 1,\ \forall y \in U \text{ with } R(x,y) \geq \alpha$$

It follows that,

$$\bigwedge_{R(x,y)\geq\alpha} \mu_A(y) + \bigvee_{R(x,y)\geq\alpha} \vartheta_A(y) \leq 1.$$

That is, $\mu_{\underline{R^\alpha}(A)}(x) + \vartheta_{\underline{R^\alpha}(A)}(x) \leq 1, \forall x \in U$.

Similarly we can prove that $\mu_{\overline{R}^\alpha(A)}(x) + \vartheta_{\overline{R}^\alpha(A)}(x) \leq 1$, $\forall x \in U$.

The authenticity of the proposed definition of α-intuitionistic fuzzy rough set is asserted by the theorem given below.

Theorem 3.1:

In the crisp case, the α-Intuitionistic fuzzy rough approximations reduces to Pawlak's rough set approximations.

Proof:

Consider a crisp approximation space (U, R). Then R can be treated as a fuzzy equivalence relation, $R: U \longrightarrow [0,1]$ as

$$R(x,y) = \chi_R(x,y) = \begin{cases} 1, & if\ y \in [x]_R \\ 0, & otherwise \end{cases}.$$

For $A \subseteq X$, the corresponding IFS is given by $A = \{(x,\ \mu_A(x),\ \vartheta_A(x)) : x \in U\}$, where $\mu_A(x) = \chi_A(x)$ and $\vartheta_A(x) = 1 - \chi_A(x)$. The Pawlak's lower and upper approximations of A are given respectively by

$$\underline{R}(A) = \{x \in U : [x]_R \subseteq A\}$$
$$\overline{R}(A) = \{x \in U : [x]_R \cap A \neq \emptyset\}.$$

The corresponding IFSs are given by

$$\mu_{\underline{R}(A)}(x) = \chi_{\underline{R}(A)}(x) = \begin{cases} 1, & if\ [x]_R \subseteq A \\ 0, & otherwise \end{cases},$$
$$\vartheta_{\underline{R}(A)}(x) = 1 - \chi_{\underline{R}(A)}(x)$$
$$\mu_{\overline{R}(A)}(x) = \chi_{\overline{R}(A)}(x) = \begin{cases} 1, & if\ [x]_R \cap A \neq \emptyset \\ 0, & otherwise \end{cases},$$
$$\vartheta_{\overline{R}(A)}(x) = 1 - \chi_{\overline{R}(A)}(x).$$

Now the proposed α-Intuitionistic fuzzy Rough Set approximations of A are given by

$$\mu_{\underline{R^\alpha}(A)} = \bigwedge_{R(x,y)\geq\alpha} \chi_A(y),$$
$$\vartheta_{\underline{R^\alpha}(A)}(x) = \bigvee_{R(x,y)\geq\alpha} (1 - \chi_A(y)),$$
$$\mu_{\overline{R}^\alpha(A)}(x) = \bigvee_{R(x,y)\geq\alpha} \chi_A(y),$$
$$\vartheta_{\overline{R}^\alpha(A)}(x) = \bigwedge_{R(x,y)\geq\alpha} (1 - \chi_A(y)).$$

Here, the only possible values of $R(x,y)$ are 0 and 1. So,

$$R(x,y) \geq \alpha \Rightarrow R(x,y) = 1 \Rightarrow y \in [x]_R.$$

Hence, $y \in A$ as $[x]_R \subseteq A$. ie; $\chi_A(y) = 1$. Thus,

$$\mu_{\underline{R^\alpha}(A)} = \bigwedge_{R(x,y)=1}(1) = 1, \text{ and}$$
$$\vartheta_{\underline{R^\alpha}(A)}(x) = \bigvee_{R(x,y)=1}(1-1) = 0.$$

When $[x]_R \nsubseteq A$, there exists an element $y \in [x]_R$ with y $\notin$ A. ie; $R(x,y) = 1$ and $\chi_A(y) = 0$. Hence,

$$\mu_{\underline{R^\alpha}(A)} = \wedge_{R(x,y)\geq\alpha} \chi_A(y) = 0, \text{ as } 1 - \chi_A(y) = 1.$$

Since $1 - \chi_A(y) = 1$,

$$\vartheta_{\underline{R^\alpha}(A)}(x) = \vee_{R(x,y)\geq\alpha}(1 - \chi_A(y) = 1.$$

Thus $\mu_{\underline{R^\alpha}(A)} = \mu_{\underline{R}(A)}$ and $\vartheta_{\underline{R^\alpha}(A)} = \vartheta_{\underline{R}(A)}$. By a similar argument, we can prove that $\mu_{\overline{R}^\alpha(A)} = \mu_{\overline{R}(A)}$ and $\vartheta_{\overline{R}^\alpha(A)} = \vartheta_{\overline{R}(A)}$.

The following proposition proves that the proposed approach provides a nested sequence of intuitionistic fuzzy rough lower and upper approximations corresponding to each IFSA on U.

Proposition 3.2:

Let A be an IFS on (U,R). Then, $\forall \alpha, \beta \in [0,1]$,

$$\alpha \leq \beta \Rightarrow \underline{R^\alpha}(A) \subseteq \underline{R^\beta}(A) \text{ and } \overline{R}^\alpha(A) \supseteq \overline{R}^\beta(A).$$

Proof:

When $\alpha \leq \beta$, $R(x,y) \geq \beta \Longrightarrow R(x,y) \geq \alpha$. So,

$$\{y \in U : R(x,y) \geq \beta\} \subseteq \{y \in U : R(x,y \geq \alpha\}.$$

It follows that,

$$\wedge_{R(x,y)\geq\alpha} \mu_A(y) \leq \wedge_{R(x,y)\geq\beta} \mu_A(y) \text{ and}$$

$$\vee_{R(x,y)\geq\alpha} \vartheta_A(y) \geq \vee_{R(x,y)\geq\beta} \vartheta_A(y).$$

ie; $\mu_{\underline{R^\alpha}(A)}(x) \leq \mu_{\underline{R^\beta}(A)}(x)$ and $\vartheta_{\underline{R^\alpha}(A)}(x) \geq \vartheta_{\underline{R^\beta}(A)}(x)$

Thus, $\underline{R^\alpha}(A) \subseteq \underline{R^\beta}(A)$. Similarly, $\overline{R}^\alpha(A) \supseteq \overline{R}^\beta(A)$.

Proposition 3.3:

Let R_1 and R_2 be two fuzzy equivalence relations on U. Then, $R_1 \subseteq R_2 \implies \underline{R_1{}^\alpha}(A) \supseteq \underline{R_2{}^\alpha}(A)$ and $\overline{R_1}^\alpha(A) \subseteq \overline{R_2}^\alpha(A)$

Proof:

$R_1 \subseteq R_2 \Longrightarrow R_1(x,y) \leq R_2(x,y)$, $\forall (x,y) \in U \times U$.

So, $R_1(x,y) \geq \alpha \Longrightarrow R_2(x,y) \geq \alpha$. Hence,

$$\{y \in U : R_1(x,y) \geq \alpha\} \subseteq \{y \in U : R_2(x,y) \geq \alpha\}.$$

It follows that,

$\mu_{\underline{R_1{}^\alpha}(A)}(x) \geq \mu_{\underline{R_2{}^\alpha}(A)}(x)$ and $\vartheta_{\underline{R_1{}^\alpha}(A)}(x) \leq \vee_{R_2(x,y)\geq\alpha} \vartheta_A(y)$

Therefore, $\underline{R_1{}^\alpha}(A) \supseteq \underline{R_2{}^\alpha}(A)$. Similarly, $\overline{R_1}^\alpha(A) \subseteq \overline{R_2}^\alpha(A)$.

Theorem 3.2:

The general properties of the α-Intuitionistic fuzzy rough approximations are give below:

(1) $\underline{R\ }(A) \subseteq A \subseteq \overline{R}^\alpha(A)$

(2) $If\ A \subseteq B$, then, $\underline{R^\alpha}(A) \subseteq \underline{R^\alpha}(B)$ and $\overline{R}^\alpha(A) \subseteq \overline{R}^\alpha(B)$

(3) $\underline{R^\alpha}(\widehat{0,1}) = (\widehat{0,1}) = \overline{R}^\alpha((\widehat{0,1}))$

(4) $\underline{R^\alpha}(\widehat{1,0}) = (\widehat{1,0}) = \overline{R}^\alpha(\widehat{1,0})$

(5) $\underline{R^\alpha}(\widehat{a,b}) = (\widehat{a,b}) = \overline{R}^\alpha(\widehat{a,b})$, $\forall a,b \in [0,1]$ with $a + b \leq 1$

(6) $\underline{R^\alpha}(1_{U-\{y\}}) \in \mathcal{F}(U)$ and $\underline{R^\alpha}(1_{U-\{y\}}) = 1_{U-\{y\}}$

(7) $\overline{R}^\alpha(1_y) \in \mathcal{F}(U)$ and $\overline{R}^\alpha(1_y) = 1_y$

Proof:

(1) Since R is a fuzzy equivalence relation, $R(x,x) = 1 \geq \alpha$, $\forall x \in X$. Hence, $x \in \{y \in U : R(x,y) \geq \alpha\}$. Therefore,

$$\wedge_{R(x,y)\geq\alpha} \mu_A(y) \leq \mu_A(x) \leq \vee_{R(x,y)\geq\alpha} \mu_A(y).$$

$$\text{ie; } \mu_{\underline{R^\alpha}(A)}(x) \leq \mu_A(x) \leq \mu_{\overline{R}^\alpha(A)}(x)$$

Also, $\vee_{R(x,y)\geq\alpha} \vartheta_A(y) \geq \vartheta_A(x) \geq \wedge_{R(x,y)\geq\alpha} \vartheta_A(y)$

$$\text{ie; } \vartheta_{\underline{R^\alpha}(A)}(x) \geq \vartheta_A(x) \geq \vartheta_{\overline{R}^\alpha(A)}(x)$$

Thus, $\underline{R^\alpha}(A) \subseteq A \subseteq \overline{R}^\alpha(A)$.

(2) $A \subseteq B \Longrightarrow \mu_A(x) \leq \mu_B(x)$, $\vartheta_A(x) \geq \vartheta_B(x)$, ∀x ∈ U. So,

$$\wedge_{R(x,y)\geq\alpha} \mu_A(y) \leq \wedge_{R(x,y)\geq\alpha} \mu_B(y)$$

$$\vee_{R(x,y)\geq\alpha} \vartheta_A(y) \geq \vee_{R(x,y)\geq\alpha} \vartheta_B(y).$$

ie; $\mu_{\underline{R^\alpha}(A)}(x) \leq \mu_{\underline{R^\alpha}(B)}(x)$ and $\vartheta_{\underline{R^\alpha}(A)}(x) \geq \vartheta_{\underline{R^\alpha}(B)}(x)$.

Hence, $\underline{R^\alpha}(A) \subseteq \underline{R^\alpha}(B)$. Similarly, $\overline{R}^\alpha(A) \subseteq \overline{R}^\alpha(B)$.

(3) We have, $(\widehat{0,1}) = \{(x,0,1) : x \in U\}$. Hence, $\forall x \in U$,

$\mu_{\underline{R^\alpha}((\widehat{0,1}))}(x) = \wedge_{R(x,y)\geq\alpha} \mu_{(\widehat{0,1})}(y) = \wedge_{R(x,y)\geq\alpha} 0 = 0$ and

$\vartheta_{\underline{R^\alpha}((\widehat{0,1}))}(x) = \vee_{R(x,y)\geq\alpha} \vartheta_{(\widehat{0,1})}(y) = \vee_{R(x,y)\geq\alpha} 1 = 1.$

Hence, $\underline{R^\alpha}(\widehat{0,1}) = (\widehat{0,1})$. Similarly, $\overline{R}^\alpha((\widehat{0,1})) = (\widehat{0,1})$.

The proof of (4) and (5) are similar.

(6) We have, $\mu_{1_{U-\{y\}}}(x) = \begin{cases} 0\ if\ x = y \\ 1\ if\ x \neq y \end{cases}$.

Since, $R(x,x) = 1 \geq \alpha$

$$\mu_{\underline{R^\alpha}(1_{U-\{y\}})}(x) = \wedge_{R(x,z)\geq\alpha} \mu_{1_{U-\{y\}}}(z) = \begin{cases} 0\ if\ x = y \\ 1\ if\ x \neq y \end{cases}.$$

Thus, $\mu_{\underline{R^\alpha}(1_{U-\{y\}})}(x) = \mu_{1_{U-\{y\}}}(x)$.

Again, $\vartheta_{1_{U-\{y\}}}(x) = \begin{cases} 1\ if\ x = y \\ 0\ if\ x \neq y \end{cases}$.

$$\vartheta_{\underline{R^\alpha}(1_{U-\{y\}})}(x) = \vee_{R(x,z)\geq\alpha} \vartheta_{1_{U-\{y\}}}(z) = \begin{cases} 1\ if\ x = y \\ 0\ if\ x \neq y \end{cases}.$$

Thus, $\vartheta_{\underline{R^\alpha}(1_{U-\{y\}})}(x) = \vartheta_{1_{U-\{y\}}}(x)$.

It follows that $\vartheta_{\underline{R^\alpha}(1_{U-\{y\}})}(x) = 1 - \mu_{\underline{R^\alpha}(1_{U-\{y\}})}(x)$.

Hence $\underline{R^\alpha}(1_{U-\{y\}}) \in \mathcal{F}(U)$. Also, $\underline{R^\alpha}(1_{U-\{y\}}) = 1_{U-\{y\}}$.

(7) The proof is smilar to that of (6).

The following theorem gives the properties of compositions of the proposed approximations.

Theorem 3.3:

The properties of the compositions of the α-Intuitionistic fuzzy rough lower and upper approximations are the following:

(8) $\underline{R^\alpha}\left(\underline{R^\alpha}(A)\right) = \underline{R^\alpha}(A) = \overline{R}^\alpha\left(\underline{R^\alpha}(A)\right)$

(9) $\underline{R^\alpha}\left(\overline{R}^\alpha(A)\right) = \overline{R}^\alpha(A) = \overline{R}^\alpha\left(\overline{R}^\alpha(A)\right)$

Proof:

(8) By property (2), $\underline{R^\alpha}\left(\underline{R^\alpha}(A)\right) \subseteq \underline{R^\alpha}(A)$. Using (5),

$$\mu_{\underline{R^\alpha}(\underline{R^\alpha}(A))}(x) = \wedge_{R(x,y)\geq\alpha} \mu_{\underline{R^\alpha}(A)}(y)$$

$$= \wedge_{R(x,y)\geq\alpha}\left(\wedge_{R(y,z)\geq\alpha} \mu_A(z)\right).$$

Consider $x \in X$. Since R is transitive, if $R(x,y) \geq \alpha$, then, $\forall z \in U$ with $R(y,z) \geq \alpha$, we get $R(x,z) \geq \alpha$. So,

$$\wedge_{R(x,u)\geq\alpha} \mu_A(u) \leq \mu_A(z).$$

ie; $\mu_{\underline{R^\alpha}(A)}(x) \leq \mu_A(z)$. So, $\mu_{\underline{R^\alpha}(A)}(x) \leq \wedge_{R(y,z)\geq\alpha} \mu_A(z)$.

This inequality is true $\forall y \in U$ with $R(x,y) \geq \alpha$. So,

$$\mu_{\underline{R^\alpha}(A)}(x) \leq \wedge_{R(x,y)\geq\alpha}\left(\wedge_{R(y,z)\geq\alpha} \mu_A(z)\right) = \mu_{\underline{R^\alpha}(\underline{R^\alpha}(A))}(x).$$

By a similar argument, $\vartheta_{\underline{R^\alpha}(A)}(x) \geq \vartheta_{\underline{R^\alpha}(\underline{R^\alpha}(A))}(x)$.

Hence, $\underline{R^\alpha}(A) \subseteq \underline{R^\alpha}(\underline{R^\alpha}(A)$. Thus, $\underline{R^\alpha}(A) = \underline{R^\alpha}(\underline{R^\alpha}(A))$.

Similarly, $\underline{R^\alpha}(A) = \overline{R}^\alpha(\underline{R^\alpha}(A))$.

(9) The Proof is similar to that of (8).

IV. INTERACTIONS WITH UNION, INTERSECTION AND COMPLEMENT

Consider a fuzzy approximation space (U, R). Let A and B be two IFSs in U. The α-Intuitionistic fuzzy rough approximations of A and B are IFSs in U. Therefore, their set theoretic operations can be defined using the corresponding operations in IFS theory as follows:

Definition 4.1:

The union of α-Intuitionistic fuzzy rough approximations are given by

$$\mu_{\underline{R^\alpha}(A)\cup\underline{R^\alpha}(B)}(x) = \max\left(\wedge_{R(x,y)\geq\alpha}\mu_A(y), \wedge_{R(x,y)\geq\alpha}\mu_B(y)\right)$$
$$\vartheta_{\underline{R^\alpha}(A)\cup\underline{R^\alpha}(B)}(x) = \min\left(\vee_{R(x,y)\geq\alpha}\vartheta_A(y), \vee_{R(x,y)\geq\alpha}\vartheta_B(y)\right).$$
$$\mu_{\overline{R}^\alpha(A)\cup\overline{R}^\alpha(B)}(x) = \max\left(\vee_{R(x,y)\geq\alpha}\mu_A(y), \vee_{R(x,y)\geq\alpha}\mu_B(y)\right)$$
$$\vartheta_{\overline{R}^\alpha(A)\cup\overline{R}^\alpha(B)}(x) = \min\left(\wedge_{R(x,y)\geq\alpha}\vartheta_A(y), \wedge_{R(x,y)\geq\alpha}\vartheta_B(y)\right)$$

Definition 4.2:

The intersection of α-Intuitionistic fuzzy rough approximations are given by

$$\mu_{\underline{R^\alpha}(A)\cap\underline{R^\alpha}(B)}(x) = \min\left(\wedge_{R(x,y)\geq\alpha}\mu_A(y), \wedge_{R(x,y)\geq\alpha}\mu_B(y)\right)$$
$$\vartheta_{\underline{R^\alpha}(A)\cap\underline{R^\alpha}(B)}(x) = \max\left(\vee_{R(x,y)\geq\alpha}\vartheta_A(y), \vee_{R(x,y)\geq\alpha}\vartheta_B(y)\right)$$
$$\mu_{\overline{R}^\alpha(A)\cap\overline{R}^\alpha(B)}(x) = \min\left(\vee_{R(x,y)\geq\alpha}\mu_A(y), \vee_{R(x,y)\geq\alpha}\mu_B(y)\right)$$
$$\vartheta_{\overline{R}^\alpha(A)\cap\overline{R}^\alpha(B)}(x) = \max\left(\wedge_{R(x,y)\geq\alpha}\vartheta_A(y), \wedge_{R(x,y)\geq\alpha}\vartheta_B(y)\right)$$

Definition 4.3:

The complement of α-Intuitionistic fuzzy Rough approximations are given by

$$\mu_{\left(\underline{R^\alpha}(A)\right)^C} = \vee_{R(x,y)\geq\alpha}\vartheta_A(y)$$
$$\vartheta_{\left(\underline{R^\alpha}(A)\right)^C}(x) = \wedge_{R(x,y)\geq\alpha}\mu_A(y).$$
$$\mu_{\left(\overline{R}^\alpha(A)\right)^C}(x) = \wedge_{R(x,y)\geq\alpha}\vartheta_A(y)$$
$$\vartheta_{\left(\overline{R}^\alpha(A)\right)^C}(x) = \vee_{R(x,y)\geq\alpha}\mu_A(y).$$

Theorem 4.1:

The α-Intuitionistic fuzzy rough lower and upper approximations are dual to each other. That is, $\forall A \in IFS(U)$,

10) $\underline{R^\alpha}(A) = \left(\overline{R}^\alpha(A^C)\right)^C$

11) $\overline{R}^\alpha(A) = \left(\underline{R^\alpha}(A^C)\right)^C$.

Proof:

(10) Let A be an IFS on U. Then,

$$\mu_{\left(\overline{R}^\alpha(A^C)\right)^C}(x) = \wedge_{R(x,y)\geq\alpha}\vartheta_{A^C}(y) = \wedge_{R(x,y)\geq\alpha}\mu_A(y) = \mu_{\underline{R^\alpha}(A)}(x).$$
$$\vartheta_{\left(\overline{R}^\alpha(A^C)\right)^C}(x) = \vee_{R(x,y)\geq\alpha}\mu_{A^C}(y) = \vee_{R(x,y)\geq\alpha}\vartheta_A(y) = \vartheta_{\underline{R^\alpha}(A)}(x).$$

It follows that $\underline{R^\alpha}(A) = (\overline{R}^\alpha(A^C))^C$.

(11) Similarly, $\overline{R}^\alpha(A) = (\underline{R^\alpha}(A^C))^C$.

Theorem 4.2:

The union and intersection of the α-Intuitionistic fuzzy rough approximations of two fuzzy sets A and B satisfy the following properties.

12) $\underline{R^\alpha}(A\cap B) = \underline{R^\alpha}(A)\cap\underline{R^\alpha}(B)$ and $\underline{R^\alpha}(\cap_i A_i) = \cap_i \underline{R}(A_i)$

13) $\overline{R}^\alpha(A\cap B) \subseteq \overline{R}^\alpha(A)\cap\overline{R}^\alpha(B)$ and $\overline{R}^\alpha(\cap_i A_i) \subseteq \cap_i \overline{R}^\alpha(A_i)$

14) $\overline{R}^\alpha(A\cup B) = \overline{R}^\alpha(A)\cup\overline{R}^\alpha(B)$ and $\overline{R}^\alpha(\cup_i A_i) = \cup_i \overline{R}^\alpha(A_i)$

15) $\underline{R^\alpha}(A\cup B) \supseteq \underline{R^\alpha}(A)\cup\underline{R^\alpha}(B)$ and $\underline{R^\alpha}(\cup_i A_i) \supseteq \cup_i \underline{R^\alpha}(A_i)$

16) $\underline{R^\alpha}\left(A\cup\widehat{(a,b)}\right) = \underline{R^\alpha}(A)\cup\widehat{(a,b)}$

17) $\overline{R}^\alpha\left(A\cap\widehat{(a,b)}\right) = \overline{R}^\alpha(A)\cap\widehat{(a,b)}$

Proof:

12) We have,

$$\mu_{\underline{R^\alpha}(A)\cap\underline{R^\alpha}(B)}(x) = \min\left(\mu_{\underline{R^\alpha}(A)}(x), \mu_{\underline{R^\alpha}(B)}(x)\right) = \min\left(\wedge_{R(x,y)\geq\alpha}\mu_A(y), \wedge_{R(x,y)\geq\alpha}\mu_B(y)\right)$$
$$\mu_{\underline{R^\alpha}(A\cap B)}(x) = \wedge_{R(x,y)\geq\alpha}\mu_{A\cap B}(y) = \wedge_{R(x,y)\geq\alpha}\min(\mu_A(y),\mu_B(y)).$$

Clearly, $\wedge_{R(x,y)\geq\alpha}\min(\mu_A(y),\mu_B(y)) \leq \wedge_{R(x,y)\geq\alpha}\mu_A(y)$ and $\wedge_{R(x,y)\geq\alpha}\min(\mu_A(y),\mu_B(y)) \leq \wedge_{R(x,y)\geq\alpha}\mu_B(y)$.

From this, it follows that,

$$\wedge_{R(x,y)\geq\alpha}\min(\mu_A(y),\mu_B(y)) \leq \min\left(\wedge_{R(x,y)\geq\alpha}\mu_A(y), \wedge_{R(x,y)\geq\alpha}\mu_B(y)\right).$$

Again, $\wedge_{R(x,y)\geq\alpha}\mu_A(y) \leq \mu_A(y)$ and $\wedge_{R(x,y)\geq\alpha}\mu_B(y) \leq \mu_B(y)$, $\forall y \in U, R(x,y)\geq\alpha$. So,

$$\min\left(\wedge_{R(x,y)\geq\alpha}\mu_A(y), \wedge_{R(x,y)\geq\alpha}\mu_B(y)\right) \leq \min(\mu_A(y),\mu_B(y))$$

This inequality is true $\forall y \in U,\ R(x,y)\geq\alpha$. Therefore,

$$\min\left(\wedge_{R(x,y)\geq\alpha}\mu_A(y), \wedge_{R(x,y)\geq\alpha}\mu_B(y)\right) \leq \wedge_{R(x,y)\geq\alpha}\min(\mu_A(y),\mu_B(y)).$$

That is, $\mu_{\underline{R^\alpha}(A)\cap\underline{R^\alpha}(B)}(x) = \mu_{\underline{R^\alpha}(A)}(x)$.

Similarly, $\vartheta_{\underline{R^\alpha}(A)\cap\underline{R^\alpha}(B)}(x) = \vartheta_{\underline{R^\alpha}(A)}(x)$.

Thus $\underline{R^\alpha}(A\cap B) = \underline{R^\alpha}(A)\cap\underline{R^\alpha}(B)$.

This can be clearly extended to $\underline{R^\alpha}(\cap_i A_i) = \cap_i A_i$.

Similarly we can prove (13), (14) and (15).

16) $$\mu_{\underline{R^\alpha}(A\cup\widehat{(a,b)})}(x) = \wedge_{R(x,y)\geq\alpha}\mu_{A\cup\widehat{(a,b)}}(y) = \wedge_{R(x,y)\geq\alpha}\max(\mu_A(y),a) = \max\left(\wedge_{R(x,y)\geq\alpha}\mu_A(y),a\right) = \max\left(\mu_{\underline{R^\alpha}(A)}(x),a\right) = \mu_{\underline{R^\alpha}(A)\cup\widehat{(a,b)}}(x).$$

Similarly, $\vartheta_{\underline{R^\alpha}(A\cup\widehat{(a,b)})}(x) = \vartheta_{\underline{R^\alpha}(A\cup\widehat{(a,b)})}(x)$.

Therefore, $\underline{R^\alpha}\left(A\cup\widehat{(a,b)}\right) = \underline{R^\alpha}(A)\cup\widehat{(a,b)}$.

17) The proof is similar to that of (16).

Remark 4.1:

Any crisp rough approximations with $\overline{R}^\alpha(A\cap B) \neq \overline{R}^\alpha(A)\cap\overline{R}^\alpha(B)$ and $\underline{R^\alpha}(A\cup B) \neq \underline{R^\alpha}(A)\cup\underline{R^\alpha}(B)$ serve as an example to show that in general, equality do not hold in properties (13) and (15) above.

The following theorem acts like a decomposition theorem for the α- intuitionistic fuzzy rough lower and upper approximations. For simplicity, the notations for the α-cuts of the fuzzy equivalence relation R and the (a,b)-cuts of the intuitionistic fuzzy set A are used to denote their membership functions (characteristic functions) also.

Theorem 4.3:

Let (U,R) be a fuzzy approximation space. Then,

18) $\overline{R}^\alpha(A) = \cup_{a,b\in I^2}\overline{{}^\alpha R}\left(A_a^b\cap\widehat{(a,b)}\right)$

19) $\underline{R^\alpha}(A) = \cap_{a,b\in I^2}\underline{{}^\alpha R}\left(A_a^b\cup\widehat{(a,b)}\right))$.

20) $\overline{R}^\alpha(A) = \cup_{a,b\in I^2}\overline{{}^\alpha R}\left(A_{a+}^{b+}\cap\widehat{(a,b)}\right)$

21) $\underline{R^\alpha}(A) = \cap_{a,b\in I^2}\underline{{}^\alpha R}\left(A_{a+}^{b+}\cup\widehat{(a,b)}\right)$,

where, $I^2 = \{(a,b)\in[0,1]\times[0,1]: a+b\leq 1\}$

Proof:

(18) Let $B = \cup_{a,b\in I^2}\overline{{}^\alpha R}(A_a^b\cap\widehat{(a,b)})$. Then,

$$\mu_B(x) = \vee_{a,b\in I^2}\mu_{\overline{{}^\alpha R}(A_a^b\cap\widehat{(a,b)})}(x)$$
$$= \vee_{a,b\in I^2}(\mu_{\overline{{}^\alpha R}(A_a^b)}(x)\wedge a)$$
$$= \vee_{a,b\in I^2}\{a : \mu_{\overline{{}^\alpha R}(A_a^b)}(x) = 1\}$$
$$= \vee_{a,b\in I^2}\{a : x \in \overline{{}^\alpha R}(A_a^b)\}$$
$$= \vee_{a,b\in I^2}\{a : [x]_{{}^\alpha R}\cap A_a^b \neq \emptyset\}$$
$$= \vee_{a,b\in I^2}\{a : \exists y,\ y\in[x]_{{}^\alpha R} \text{ and } y \in A_a^b\}$$
$$= \vee_{a,b\in I^2}\{a : \exists y,\ R(x,y)\geq\alpha, \mu_A(y)\geq a, \vartheta_A(y)\leq b\}$$
$$= \vee_{R(x,y)\geq\alpha}\mu_A(y) = \mu_{\overline{R}^\alpha(A)}(x).$$

$\vartheta_B(x) = \bigwedge_{a,b\in I^2} \vartheta_{\overline{^{\alpha}R}(A_a^b \cap \widehat{(a,b)})}(x)$
$= \bigwedge_{a,b\in I^2}(\vartheta_{\overline{^{\alpha}R}(A_a^b)}(x) \vee b)$
$= \bigwedge_{a,b\in I^2}\{b : \vartheta_{\overline{^{\alpha}R}(A_a^b)}(x) = 0\}$
$= \bigwedge_{a,b\in I^2}\{b : x \in \overline{^{\alpha}R}(A_a^b)\}$
$= \bigwedge_{a,b\in I^2}\{b : [x]_{\alpha_R} \cap A_a^b \neq \emptyset\}$
$= \bigwedge_{a,b\in I^2}\{b : \exists y,\ y \in [x]_{\alpha_R}$ and $y \in A_a^b\}$
$= \bigwedge_{a,b\in I^2}\{b : \exists y, R(x,y) \geq \alpha$ and $\mu_A(y) \geq a, \vartheta_A(y) \leq b\}$
$= \bigwedge_{R(x,y)\geq\alpha} \vartheta_A(y) = \vartheta_{\overline{R}^{\alpha}(A)}(x)$.

Similarly (19) can be proved.

(20) Let $C = \bigcap_{a,b\in I^2} \underline{^{\alpha}R}\left(A_a^b \cup \widehat{(a,b)}\right)$. Then,

$\mu_C(x) = \mu_{\bigcap_{a,b\in I^2} \underline{^{\alpha}R}(A_a^b \cup \widehat{(a,b)})}(x)$
$= \bigwedge_{a,b\in I^2} \mu_{\underline{^{\alpha}R}(A_a^b \cup \widehat{(a,b)})}(x)$
$= \bigwedge_{a,b\in I^2}(\mu_{\underline{^{\alpha}R}(A_a^b)}(x) \vee a)$
$= \bigwedge_{a,b\in I^2}\left\{a : \mu_{\underline{^{\alpha}R}(A_a^b)}(x) = 0\right\}$
$= \bigwedge_{a,b\in I^2}\left\{a : x \notin \underline{^{\alpha}R}(A_a^b)\right\}$
$= \bigwedge_{a,b\in I^2}\{a : [x]_{\alpha_R} \not\subseteq A_a^b\}$
$= \bigwedge_{a,b\in I^2}\{a : \exists y, y \in [x]_{\alpha_R}, y \notin A_a^b\}$
$= \bigwedge_{a,b\in I^2}\{a : \exists y,\ R(x,y) \geq \alpha$ but $\mu_A(y) < a\ or\ \vartheta_A(y) > b\}$
$= \bigwedge_{R(x,y)\geq\alpha} \mu_A(y) = \mu_{\underline{R^{\alpha}}(A)}(x)$.

$\vartheta_C(x) = \vartheta_{\bigcap_{a,b\in I^2} \underline{^{\alpha}R}(A_a^b \cup \widehat{(a,b)})}(x)$
$= \bigvee_{a,b\in I^2} \vartheta_{\underline{^{\alpha}R}(A_a^b \cup \widehat{(a,b)})}(x)$
$= \bigvee_{a,b\in I^2}(\vartheta_{\underline{^{\alpha}R}(A_a^b)}(x) \wedge b)$
$= \bigvee_{a,b\in I^2}\{b : \vartheta_{\underline{^{\alpha}R}(A_a^b)}(x) = 1\}$
$= \bigvee_{a,b\in I^2}\left\{b : x \notin \underline{^{\alpha}R}(A_a^b)\right\} = \bigvee_{a,b\in I^2}\{b : [x]_{\alpha_R} \not\subseteq A_a^b\}$
$= \bigvee_{a,b\in I^2}\{b : \exists y,\ y \in [x]_{\alpha_R},\ y \notin A_a^b\}$
$= \bigvee_{a,b\in I^2}\{b : \exists y,\ R(x,y) \geq \alpha$ but $\mu_A(y) < a\ or\ \vartheta_A(y) > b\}$
$= \bigwedge_{R(x,y)\geq\alpha} \vartheta_A(y) = \vartheta_{\overline{R}^{\alpha}(A)}(x)$.

Similarly (21) can be proved.

V. TOPOLOGICAL AND LATTICE THEORETICAL PROPERTIES

In this section, some topological and lattice theoretical properties of α-intuitionistic fuzzy rough sets are discussed. Every fuzzy equivalence relation R on U induces an intuitionistic fuzzy topology on U in which the intuitionistic fuzzy rough lower and upper approximations act as the interior and closure operators respectively. It also determines three complete distributive lattices of intuitionistic fuzzy subsets of U.

Theorem 4.1: Consider a fuzzy approximation space (U,R). Let $\tau = \left\{A \in IFS(U): \underline{R^{\alpha}}(A) = A\right\}$. Then τ is an intuitionistic fuzzy topology on U.

Proof:

By property (3) and (4), ($\widehat{(0,1)} \in \tau$ and $\widehat{(1,0)} \in \tau$. Also, $A, B \in \tau \Longrightarrow \underline{R^{\alpha}}(A) = A,\ \underline{R^{\alpha}}(B) = B$.

Hence $\underline{R^{\alpha}}(A \cap B) = \underline{R^{\alpha}}(A) \cap \underline{R^{\alpha}}(B) = A \cap B$ and $A \cap B \in \tau$.

For $A_i \in \tau,\ \underline{R^{\alpha}}(A_i) = A_i$. Clearly, $\underline{R^{\alpha}}(\bigcup_i A_i) \subseteq \bigcup_i A_i$. By property (15), $\underline{R^{\alpha}}(\bigcup_i A_i) \supseteq \bigcup_i \underline{R^{\alpha}}(A_i) = \bigcup_i A_i$. Therefore, $\underline{R^{\alpha}}(\bigcup_i A_i) = \bigcup_i A_i$. Hence, $\bigcup_i A_i \in \tau$.

Thus $\tau = \left\{A \in IFS(U): \underline{R^{\alpha}}(A) = A\right\}$ is an intuitionistic fuzzy topology on U.

Corollary 4.1: $\underline{R^{\alpha}}$ and $\overline{R}^{\alpha}$ are respectively the IF interior operator and IF closure operator in (U,τ).

Proof:

For $A \in IFS(U)$, $int(A) = \bigcup\{G: G \in \tau\ and\ G \subseteq A\}$. Now, $G \in \tau \Longrightarrow \underline{R^{\alpha}}(G) = G$. Also $G \subseteq A \Longrightarrow \underline{R^{\alpha}}(G) \subseteq \underline{R^{\alpha}}(A)$. So, $G \subseteq \underline{R^{\alpha}}(A)$ and $int(A) = \bigcup G \subseteq \underline{R^{\alpha}}(A)$. By property (8), $\underline{R^{\alpha}}(A) \in \tau$. Further, $\underline{R^{\alpha}}(A) \subseteq A$. Hence, $\underline{R^{\alpha}}(A) \subseteq \bigcup\{G: G \in \tau\ and\ G \subseteq A\} = int(A)$. Thus $\underline{R^{\alpha}}(A) = int(A)$. Similarly, $cl(A) = \overline{R}^{\alpha}(A)$.

It may be recalled that a lattice $(L, \leq, \vee, \wedge)$ consists of a non-empty set L, a partial order (reflexive, anti-symmetric and transitive relation) $\leq$ and two binary operations $\vee$and $\wedge$ called join and meet respectively. L is called a complete lattice if $a \vee b$ and $a \wedge b$ exists $\forall a, b \in L$. L is called a distributive lattice if (1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ and (2) $(a \wedge b) \vee c = (a \vee c) \wedge (a \vee c)$.

Theorem 4.2: Let $\tau = \left\{A \in IFS(U): \underline{R^{\alpha}}(A) = A\right\}$ and $\rho = \left\{A \in IFS(U): \overline{R}^{\alpha}(A) = A\right\}$. Then $(\tau, \subseteq, \cup, \cap)$ and $(\rho, \subseteq, \cup, \cap)$ are complete distributive lattices with $\widehat{(0,1)}$ as the least element and $\widehat{(1,0)}$ as the greatest element.

Proof:

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x),\ \vartheta_A(x) \geq \vartheta_B(x), \forall x \in U$$

defines a partial order on both τ and ρ.

For $A_i \in \tau$, $\underline{R^{\alpha}}(A_i) = A_i$. Using properties (12) and (15), $\underline{R^{\alpha}}(\bigcap_i A_i) = \bigcap_i A_i$ and $\underline{R^{\alpha}}(\bigcup_i A_i) \supseteq \bigcup_i \underline{R^{\alpha}}(A_i) = \bigcup_i A_i$.

Also, $\underline{R^{\alpha}}(\bigcup_i A_i) \subseteq \bigcup_i A_i$. Therefore, $\underline{R^{\alpha}}(\bigcup_i A_i) = \bigcup_i A_i$. Hence, $\bigcap_i A_i \in \tau, \bigcup_i A_i \in \tau$. Thus τ is a complete lattice.

By the properties of the union and intersection of intuitionistic fuzzy sets, it follows that τ is a complete distributive lattice. Also, $\widehat{(0,1)} \in \tau$ and $\widehat{(1,0)} \in \tau$. It is obvious that $\widehat{(0,1)}$ is the least and $\widehat{(1,0)}$ is the greatest element of τ.

For $A_i \in \tau$, $\overline{R}^{\alpha}(A_i) = A_i$. By properties (13) and (14), $\overline{R}^{\alpha}(\bigcup_i A_i) = \bigcup_i A_i$ and $\overline{R}^{\alpha}(\bigcap_i A_i) \subseteq \bigcap_i \overline{R}^{\alpha}(A_i) = \bigcap_i A_i$.

Clearly, $\bigcap_i A_i \subseteq \overline{R}^{\alpha}(\bigcap_i A_i)$. Hence $\overline{R}^{\alpha}(\bigcap_i A_i) = \bigcap_i A_i$. Thus $\bigcap_i A_i \in \rho,\ \bigcup_i A_i \in \rho$. Thus ρ is a complete lattice.

By the properties of the union and intersection of intuitionistic fuzzy sets, it follows that ρ is a complete distributive lattice. Also, $\widehat{(0,1)} \in \rho$ and $\widehat{(1,0)} \in \rho$. Thus, $\widehat{(0,1)}$ is the least and $\widehat{(1,0)}$ is the greatest element of ρ.

Corollary 4.2: The family $\left\{A \in IFS(U): \underline{R^{\alpha}}(A) = A = \overline{R}^{\alpha}(A)\right\}$ of all definable sets on U is a complete distributive lattice with $\widehat{(0,1)}$ as the least element and $\widehat{(1,0)}$ as the greatest element.

VI. CONCLUSION

Rough set theory, fuzzy set theory and IFS theory deal with sets having imprecise boundaries. The contemporary concern about unravelling hidden knowledge from imperfect data and incomplete information systems has lead to many hybrid theories which combine rough set theory with fuzzy set theory and IFS theory. The definition of fuzzy rough set proposed by A. Nakamura [18], is a simple and straightforward generalization of a rough set. However, not much work has been done in this direction. In this paper, the concept of α-intuitionistic fuzzy rough sets has been introduced by extending this definition of fuzzy rough sets into the intuitionistic fuzzy context. Each IFS A on U has been provided with a nested sequence of α-intuitionistic fuzzy rough lower and upper approximations. The properties of α-intuitionistic fuzzy rough approximations were investigated. It has been verified that the α-intuitionistic fuzzy rough approximations coincide with Pawlak's rough set approximations in the crisp case. A decomposition theorem for the α-intuitionistic fuzzy rough lower and upper approximations in terms of the α-cuts of the fuzzy equivalence relation and the (a,b)-cuts of the intuitionistic fuzzy set has been presented. Further it has been proved that every fuzzy equivalence relation R on U induces an intuitionistic fuzzy topology on U in which the α-intuitionistic fuzzy rough lower and upper approximations act as the interior and closure operators respectively and three complete distributive lattices of intuitionistic fuzzy subsets of U. Future work involves application of α-intuitionistic fuzzy rough sets in image processing, especially in object- background classification and exploring attribute reduction using α -intuitionistic fuzzy rough sets which may be applied in data mining.

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