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On the Oscillation of a Class of Second Order Nonlinear Neutral Delay Difference Equations

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Abstract: In this paper, second order nonlinear neutral difference equations with delay are considered. The aim of this paper is to obtain sufficient conditions for the oscillation of solutions of the equations considered. The results obtained in this paper are supported with suitable examples and numerical simulations are provided.

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Keywords **:** Oscillation, neutral, delay second order, difference equation.

I. INTRODUCTION

The study of oscillation of difference equations has received great attention and has been studied extensively. The reader is referred to the monographs [1, 2], [12] and papers [3] – [10], [14] – [21] and the references cited therein. Neutral difference equations find numerous applications in natural science and technology. They are frequently used for the study of distributed networks containing lossless transmission lines. Recently much research have been carried out on the oscillatory and asymptotic behavior of delay and neutral delay type difference equations. The main purpose of this paper is to find oscillation criteria for second order non linear difference equation of the form

$$\Delta^2\left[x(n)+p(n)x(n-\tau)\right] +q(n+1)f\left(x(n+1-\sigma)\right)=0, n\geq 0 \qquad (1)$$

Where $\tau,\sigma>0, p(n)$ and $q(n)$ are real valued sequences and $f\in C(R:R)$.

In this paper, we consider the following cases

$(H_1)-\mu\leq p(n)\leq 0$ where $\mu\in(0,1)$

$(H_2)0\leq p(n)<\infty$.

Throughout this paper, we assume that equation (1) satisfies the following hypothesis;

$(H_3)\dfrac{f(x)}{x}\geq\lambda>0 \; for \; x\neq 0;$

$(H_4)q(n)\geq M>0 \; for \; n\geq 0.$

We shall use the following notations: $N=\{0,1,2,\ldots\}$ the set of natural numbers including zero and $N(k)=\{k,k+1,k+2,\ldots\}$ where $k\in N$.

By a solution of (1), we mean any function $x(n):Z\to R$, (Z and R are set of integers and real numbers respectively) which is defined for $n\geq\min(\tau-\sigma)$ and satisfies (1) for sufficiently large n. We consider only such solutions which are non trivial for all large n.

As it is customary, a solution of equation (1) is said to be oscillatory if the terms *x(n)* of the solution are neither eventually positive nor eventually negative. Otherwise, the solution is said to be non-oscillatory.

In [20, 21], the authors investigated the oscillatory behavior of higher order and even order neutral difference equation when $p(n)$ is an oscillatory sequence. The papers [3, 7, 8, 13, 16, 17] are devoted to the case $0\leq p(n)<1$. In [3], the authors also discussed the case $-1<p(n)\leq 0$. The papers [6, 9, 14, 18, 19] deal with the oscillation and non oscillation of difference equations of a variety of classes under the assumption *p(n)* is a non negative sequence. The paper [11] deals with the oscillation and nonoscillation of a non linear neutral delay difference equations when $0\leq p(n)<1$ and $-\mu\leq p(n)<0$.

II. PRELIMINARY LEMMAS

In this section we state Lemmas and Theorems which play crucial role in proving the results in section 3.

Definition: The linear space $l_p, 1\leq p<\infty$ of all sequences $x=\{x_1,x_2,\ldots,x_n,\ldots\}$ of scalars such that $\sum_{n=1}^{\infty}|x_n|^p<\infty$ with the norm $\|x\|=\left[\sum_{n=1}^{\infty}|x_n|^p\right]^{\frac{1}{p}}$ is a Banach space.

We will compare the following inequalities

$$\Delta\left[a(n)\Delta x(n)\right]+q(n)f\left(x(\sigma(n))\right)\leq 0 \qquad (E_1)$$

$$\Delta\left[a(n)\Delta x(n)\right]+q(n)f\left(x(\sigma(n))\right)\geq 0 \qquad (E_2)$$

With the equation

$$\Delta\left[a(n)\Delta x(n)\right]+q(n)f\left(x(\sigma(n))\right)=0 \qquad (E_3)$$

The following Lemmas are taken from [4]

Lemma A.[[4], Lemma 2.1]: If inequality (E_1) (inequality (E_2)) has an eventually positive (negative) solution, then equation (E_3) also has eventually positive (negative) solution.

Lemma B. [[4], Lemma 2.1]: If the inequality

$$\Delta x(n)+q(n)f\left(x\left[\sigma(n)\right]\right)\leq 0$$

$$or\ \Delta x(n)-q(n)f\left(x\left[\sigma(n)\right]\right)\geq 0$$

where $\sigma(n), q(n)$ and $f(x)$ satisfy conditions given in (H_4) has eventually positive solution, then the equation

$$\Delta x(n)+q(n)f\left(x\left[\sigma(n)\right]\right)=0$$

$$or\ \Delta x(n)-q(n)f\left(x\left[\sigma(n)\right]\right)=0$$

also has eventually positive solution.

Theorem C.[5]: Assume that (H_4) holds and $p(n)\geq 0$ eventually. Furthermore, assume that

$$\limsup_{n\to\infty}\sum_{i=\sigma(n)}^{n-1}p(i)<\infty$$

$$\limsup_{n\to\infty}\sum_{i=\sigma(n)}^{n-1}p(i)>\frac{1}{e}.$$

Then all solutions of

$$\Delta x(n)+p(n)x(\sigma(n))=0, n=0,1,2,...$$

are oscillatory.

In [14], the authors considered the delay difference equation of the form

$$y_{n+1}-y_n+p_n y_{n-k}=0, n=0,1,2,... \qquad (2)$$

where $\{p_n\}$ is a sequence of non negative real number and k is a positive integer and proved the following theorem .

Theorem D. [14], (Theorem 2)]: If for large n,

$$\sum_{i=n-k}^{n}p_i\leq\frac{1}{e}$$

then (3) has a non oscillatory solution.

III. THE CASE

$$-1\leq -\mu\leq p(n)\leq 0$$

Theorem 1. Assume that $\sigma>2\tau$ and (H_1), (H_3) hold. Assume further that *f(x)* is non decreasing and

$$(H_5)\sum_{i=n_0}^{\infty}q(i+1)\geq\frac{\mu}{\lambda e\tau}.$$

Then equation (1) is oscillatory.

Proof. Assume that equation (1) has a non oscillatory solution *x(n)*. Without loss of generality, we can assume that *x(n)* is an eventually positive solution (the proof is similar when *x(n)* is eventually negative). Then *x(n)* for $n\geq n_0-\sigma>0$.

Define a sequence $z(n)=x(n)+p(n)x(n-\sigma)$.

Then from equation (1) we obtain

$$\Delta^2 z(n)+q(n+1)f\left(x(n+1-\sigma)\right)=0$$

$$\Delta^2 z(n)=-q(n+1)f\left(x(n+1-\sigma)\right)\leq 0$$

Hence we obtain

$$\Delta\left(\Delta z(n)\right)\leq 0 \qquad (3)$$

which implies that there exists $n_1\in N(n_0)$ such that $\Delta z(n)>0$ or $\Delta z(n)<0$ for $n\in N(n_1)$.

Case (i) : If $\Delta z(n)>0$ for $n\geq n_1$, then there exists $n_2>n_1$ such that $z(n)>0$ or $z(n)<0$ for $n\geq n_2-\sigma$.

Let us first consider the case $\Delta z(n)>0$ and $\Delta z(n)<0$ for $n\geq n_2-\sigma$. Since $x(n-\delta)\geq z(n-\sigma)$ and $f(u)$ is non increasing, it follows from (3) that

$$\Delta^2 z(n)+q(n+1)f\left(z(n+1-\sigma)\right)\leq 0, n\in N(n_2)$$

$$\Delta^2 z(n)\leq -q(n+1)f\left(z(n+1-\sigma)\right)\leq 0, n\in N(n_2) \quad (4)$$

Summing the above inequality (4) from n_2 to n - 1, we obtain

$$\sum_{j=n_2}^{n-1}\Delta\left(\Delta z(j)\right)\leq -\sum_{j=n_2}^{n-1}q(j+1)f\left(z(j+1-\sigma)\right)$$

$$\Delta z(n)-\Delta z(n_2)\leq -\sum_{j=n_2}^{n-1}q(j+1)f\left(z(j+1-\sigma)\right)$$

$$\leq f\left(z(n_2-\sigma)\right)\sum_{j=n_2}^{n-1}q(j+1)$$

Thus

$$\infty>\Delta z(n_2)\geq\Delta z(\infty)+f\left(z(n_2-\sigma)\right)\sum_{j=n_2}^{\infty}q(j+1)$$

$$\geq f\left(z(n_2-\sigma)\right)\sum_{j=n_2}^{\infty}q(j+1)\to\infty$$

which is a contradiction.

Next, we claim that the case $\Delta z(n)>0$ and $z(n)<0$ for $n\geq n_2-\sigma$ does not hold. Otherwise, Since $x(n-\sigma)>-\frac{1}{\mu}z(n+\tau-\sigma)$ and *f(u)* is non decreasing, it follows from (3) that

$$\Delta^2 z(n)+q(n+1)f\left(-\frac{1}{\mu}z(n+\tau-\sigma)\right)\leq 0, \quad (5)$$

$$n\in N(n_2)$$

Summing the above inequality (5) from n to $n+\tau-1$, we obtain

$$\sum_{s=n}^{n+\tau-1}\Delta\left(\Delta z(s)\right)+\sum_{s=n}^{n+\tau-1}q(s+1)f\left(-\frac{1}{\mu}z(s+\tau-\sigma)\right)\leq 0$$

$$\Delta z(n+\sigma)-\Delta z(n)+\sum_{s=n}^{n+\tau-1}q(s+1)f\left(-\frac{1}{\mu}z(s+\tau-\sigma)\right)\leq 0$$

$$-\Delta z(n)+\sum_{s=n}^{n+\tau-1}q(s+1)f\left(-\frac{1}{\mu}z(s+\tau-\sigma)\right)\leq -\Delta z(n+\tau)<0$$

$$-\Delta z(n)+\sum_{s=n}^{n+\tau-1}q(s+1)f\left(-\frac{1}{\mu}z(s+\tau-\sigma)\right)<0$$

Since $-\frac{1}{\mu}z(n)$ is increasing for $n\geq n_2$, we have

$$-\Delta z(n)+f\left(-\frac{1}{\mu}z(n+2\tau-\sigma)\right)\sum_{s=n}^{n+r-1}q(s+1)<0$$

By condition (H_3) we have

$$-\Delta z(n)-z(n+2\tau-\sigma)+f\left(-\frac{1}{\mu}z(s+\tau-\sigma)\right)\sum_{s=n}^{n+r-1}q(s+1)<0$$

That is

$$\Delta z(n)+z(n+2\tau-\sigma)\frac{\lambda}{\mu}\sum_{s=n}^{n+\tau-1}q(s+1)<0 \quad (6)$$

Applying Theorem C with $a(n)=\frac{\lambda}{\mu}\sum_{s=n}^{n+\tau-1}q(s+1)$, by ($H_5$) we have

$$\sum_{s=n-\tau}^{n-1}a(s)\geq\frac{1}{e}.$$

Therefore inequality (6) has no eventually negative solution which contradicts the assumption that $z(n)<0$ *for* $n\geq n_2-\sigma$.

Case (ii) Let $\Delta z(n)<0$ *for* $n\in N(n_1)$.

The inequality $\Delta z(n)\leq\Delta z(n_1)<0$ implies that

$$\lim_{n\to\infty}z(n)=-\infty$$

Then there exists c > 0 and $n_3\in N(n_0)$ such that

$$x(n)+p(n)x(n-\tau)\leq -\text{c},\ \text{n}\in \text{N}(n_3)$$

which yields

$$x(n+\tau)\leq -\text{c}+\mu \text{x}(n),\ \text{n}\in\text{N}(n_3)$$

Continuing inductively, we get

$$x(n_3+i\tau)\leq-\sum_{j=1}^{n}c\mu^{j-1}+\mu^i x(n_3).$$

Therefore $x(n_3+i\tau)<0$ for sufficiently large n which contradicts the assumption that $x(n)>0$ for $n\in N(n_1)$. The proof is complete.

Theorem 2. Suppose that (H_1),(H_3) and (H_4) hold, then every solution of equation (1) either oscillates or tends to zero as $n\to\infty$.

Proof. Suppose that equation (1) has non oscillatory solution. Without loss of generality we can assume that x(n) is a positive solution. Hence we have $x(n)>0$ for $n\geq n_0-\sigma>0$ is a solution of equation (1)

Define the sequence $z(n)=x(n)+p(nx(n-\tau))$.

Then from equation (1) we see that

$$\Delta^2 z(n)=-q(n+1)f(x(n+1-\sigma))\leq 0 \qquad (7)$$

which implies that there exists $n_1\in N(n_0)$ such that $\Delta z(n)>0$ or $\Delta z(n)=0$ for $n\in N(n_1)$. We claim that the case $\Delta z(n)>0$ for $n\in N(n_1)$ does not hold.

Otherwise summing (7) from n_1 *to* $n-1$, we obtain

$$\sum_{j=n_1}^{n-1}\Delta[\Delta z(j)]+\sum_{j=n_1}^{n-1}q(j+1)f(x(j+1-\sigma))\leq 0$$

$$\Delta z(n)-\Delta z(n_1)\leq-\sum_{j=n_1}^{n-1}q(j+1)f(x(j+1-\sigma))$$

$$\Delta z(n_1)-\Delta z(n)\geq\sum_{j=n_1}^{n-1}M\lambda x(j+1-\sigma)$$

$$\Delta z(n_1)-\Delta z(n_1)\geq \text{M}\lambda\sum_{j=n_1}^{n-1}x(j+1-\sigma)$$

Thus we obtain

$$M\lambda\sum_{j=n_1}^{n-1}x(j+1-\sigma)\leq\Delta z(n_1)-\Delta z(n)\leq\Delta z(n_1)<\infty$$

which implies that $x(n)\in l_1$

Then $z(n)=x(n)+p(n)x(n-\tau)\in l_1$

Since *z(n)* is increasing and then $\lim_{n\to\infty}z(n)=0$

We claim that *x(n)* is bounded for $n\geq n_1$.

If not, then $\limsup_{n\to\infty}x(n)=\infty$ so that there exists a sequence $n_1\in N(n_1)$ satisfying

$x(n_i)\geq x(n_i-\tau)$ *and* $x(n_i)\to\infty$ *as* $n\to\infty$. Then we have

$$z(n_i)=x(n_i)+p(n_i)x(n_i-\tau)\geq x(n_i)-\mu x(n_i-\tau)$$
$$=(1-\mu)x(n_i)\to\infty, i\to\infty$$

which contradicts the fact that $\lim_{n\to\infty}z(n)=0$. From this claim, we conclude that there exists $c\geq 0$ such that $\limsup_{n\to\infty}x(n)=c$.

Then

$$\begin{aligned}0&=\limsup_{n\to\infty}\left[x(n)+p(n)x(n-\tau)\right]\\&\geq\limsup_{n\to\infty}\left[x(n)-\mu x(n-\tau)\right]\\&\geq\limsup_{n\to\infty}\left[x(n)\right]+\liminf_{n\to\infty}\left[-\mu x(n-\tau)\right]\\&=\limsup_{n\to\infty}x(n)-\liminf_{n\to\infty}\left[-\mu x(n-\tau)\right]\\&=(1-\mu)c\end{aligned}$$

Therefore $\limsup_{n\to\infty}x(n)=c=0$.

Hence $\limsup_{n\to\infty}x(n)=0$.

The proof for the case $\Delta z(n)<0$ for $n\in N(n_1)$ does not hold is similar to the proof in Theorem 1. The proof is complete

IV. THE CASE

$0\leq p(n)<\infty$

Theorem 3. Suppose that (H_2) – (H_4) hold. Then equation (1) is oscillatory.

Proof. Suppose the equation (1) has a nonoscillatory solution of (1). Without loss of generality we can assume that *x(n)* is an eventually positive solution of (1). Hence we have $x(n)>0$ for $n\geq n_0-\sigma>0$ is a solution of equation (1).

Define $z(n)=x(n)+p(n)x(n-\tau)$.

Then $z(n)>0$. It follows from equation (1) that

$$\Delta^2 z(n)=-q(n+1)f(x(n+1-\sigma))\leq 0$$

This implies that there exists $n_1\in N(n_0)$ such that $\Delta z(n)>0$ or $\Delta z(n)<0$ for $n\in N(n_1)$.

Then proceeding as in the Theorem 2, we get $z(n)=x(n)+p(n)x(n-\tau)\in l_1$.

But the inequality $z(n)\geq z(n_1)>0$ implies that $z(n)\notin l_1$ which is a contradiction.

If $\Delta z(n)<0$ *for* $n\geq n_0$, then $\lim_{n\to\infty}z(n)=-\infty$

which contradicts the fact that $z(n) > 0$ for $n \geq n_0 - \sigma > 0$. The proof is complete.

V. EXAMPLES

In this section, we demonstrate the results obtained for the oscillation of equation (1). Numerical simulations with MATLAB are provided to support the results.

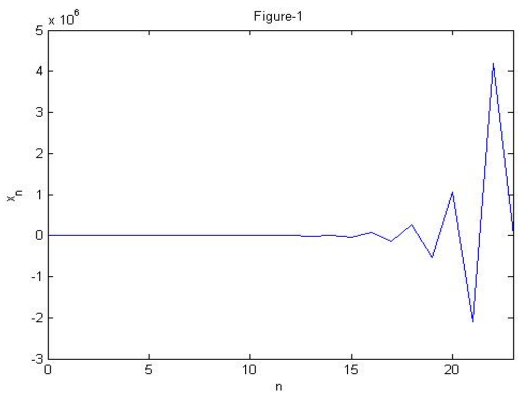
Example 1. For the difference equation

$$\Delta^2\left[x(n) - \frac{1}{2}x(n-1)\right] + \frac{45}{2^{2n-7}}x^3(n-3) = 0, n > 3 \quad (8)$$

we have $p(n) = -\frac{1}{2},\ \mu = \frac{1}{2,}$, $q(n) - \frac{45}{4}2^{\frac{2(n+1)}{3}}$

$\tau = 1, \sigma = 4$ and $f(u) = u^{\frac{1}{3}}$. Also $\sum_{i=4}^{\infty} q(i+1) = 120 \geq \frac{\mu}{ke\tau}$.

Hence, by Theorem (1), equation (8) is oscillatory.



Graph of solution of Equation (8)

Example 2. We consider the difference equation

$$\Delta^2\left[x(n) - \frac{1}{2}x(n-1)\right] + 9 \times 2^{2(n-2)}x^3(n-1) = 0, n > 1 \quad (9)$$

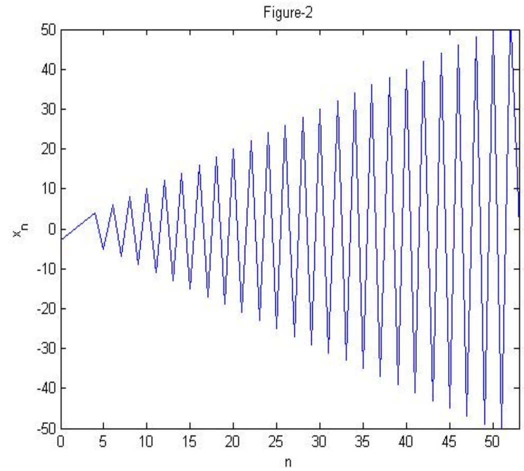
$p(n) = -\frac{1}{2}, q(n) = 9 \times 2^{2(n-2)}$ and $f(u) = u^3$. Hence by Theorem (2), equation (9) has a solution which tends to zero or oscillatory. One such oscillatory solution is $x(n) = \frac{(-1)^n}{2^n}$.

Example 3. For the difference equation

$$\Delta^2\left[x(n) + \frac{1}{n-1}x(n-1)\right] + 4\frac{n}{n-3}x(n-3) = 0, n > 3 \quad (10)$$

$p(n) = \frac{1}{n-1} > 0 \ for\ n > 3,\ q(n) = 4\frac{n}{n-3}$ and $f(u) = u$.

By theorem (3), the equation (10) is oscillatory.



Graph of solution of Equation (10)

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