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Hybrid Events Joint Entropy and Conditional Entropy Based on Chance Measure

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Abstract: We usually meet many uncertain phenomena because "uncertainty is absolute and certainty is relative" in the real world. Many researchers have studied the entropy of many events for measuring the degree of their uncertainty. In this paper, we study the joint entropy of hybrid vector and the conditional entropy of a hybrid variable given another, and give the form of conditional entropy with discrete random fuzzy variable and discrete fuzzy random variable.

Keywords: Hybrid variable; Chance measure; Joint entropy; Conditional entropy

I. INTRODUCTION

We usually meet many uncertain phenomena because "uncertainty is absolute and certainty is relative" in the real world. In these uncertain events, fuzziness and randomness are two basic types of uncertainty. Probability theory is a branch of mathematics for studying the behavior of random phenomena. The study of probability theory was started by Pascal and Fermat (1654), and an axiomatic foundation of probability theory was given by Kolmogoroff (1933) in his *Foundations of Probability Theory.* Credibility theory is a branch of mathematics for studying the behavior of fuzzy phenomena. The study of credibility theory was started by Liu and Liu (2002), and an axiomatic foundation of credibility theory was given by Liu (2004) in his *Uncertainty Theory*. Sometimes, fuzziness and randomness simultaneously appear in a system. In order to describe this phenomena, a hybrid variable was introduced by Liu [10] as a tool to describe the quantities with fuzziness and randomness. Fuzzy random variable and random fuzzy variable are instances of hybrid variable. In order to measure hybrid events, a concept of chance measure was introduced by Li and Liu [14].

Entropy is use to provide a quantitative measurement of the degree of uncertainty, which has widely been applied in Transportation [15]&[16], risk analysis [18], signal processing [17] and economics [19]. Since the Shannon entropy of random variables was proposed by Shannon [4], Jaynes [12] provided the maximum entropy principle of random variables when some constraints were given. Fuzzy entropy was first initialized by Zadeh [5] to quantify the fuzziness, who defined the entropy of a fuzzy event as weighted Shannon entropy. Up to now, fuzzy entropy has been studied by many researchers such as De Luca and Termini [2], Kaufmann [3], Yager [14], Kosko [13], Pal and Pal [6], Bhandari and Pal [1], Pal and Bezdek [7]. However, those definitions of entropy characterize the uncertainty resulting primarily from the linguistic vagueness rather than resulting from information deficiency, and vanish when the fuzzy variable is an equipossible one. In order to measure the uncertainty of fuzzy variables, Liu [9] suggested that entropy of fuzzy variables should meet at least three basic requirements: (i) minimum; (ii) maximum; (iii) universality. In order to meet those requirements, Li and Liu [8] provided a new definition of fuzzy entropy to characterize the uncertainty resulting from information deficiency which is caused by the impossibility to predict the specified value that a fuzzy variable takes. In order to measure the uncertainty of hybrid variables, Li X, and Liu B [11] provided the concept of hybrid entropy. So, on the basis of their work, we study the joint entropy for hybrid vectors and the conditional entropy for a hybrid variable given another, and give the form of conditional entropy with discrete random fuzzy variable and discrete fuzzy random variable in this paper.

The organization of our work is as follows: In section 2, some basic concepts and results are reviewed. In section 3, we introduce the joint entropy of hybrid vector. In sections 4, we introduce the conditional entropy of a hybrid variable given another, and give the form of conditional entropy with discrete random fuzzy variable and discrete fuzzy random variable. Finally, the conclusion is given in the last section

II. PRELIMINARIES

Fuzzy set theory has been well developed and applied in a wide variety of real problems. Let ξ be a fuzzy variable with membership function u and B a set of real numbers. Then the possibility, necessity, and credibility measure of fuzzy event $\xi \in B$ can be represented by

$$\mathrm{Pos}\{\xi \in B\} = \sup_{x \in B} u(x),$$

$$\mathrm{Nec}\{\xi \in B\} = 1 - \sup_{x \notin B} u(x),$$

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\mathrm{Pos}\{\xi \in B\} + \mathrm{Nec}\{\xi \in B\}\right).$$

Let ξ be a fuzzy variable with the membership function $\mu(x)$ which satisfies the normalization condition, i.e., $\sup_x \mu(x) = 1$. In the setting of credibility theory, the credibility measure for fuzzy event $\{\xi \in B\}$ deduced from $\mu(x)$ is given by

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)\right)$$

Where B is any subset of the real numbers R, and B^c is the complement of set B. Conversely, for a fuzzy variable ξ, its membership function can be derived from the credibility measure by

$$\mu(x) = \left(2\mathrm{Cr}\{\xi = x\}\right) \wedge 1, \qquad x \in R$$

Definition A. A hybrid variable is a measurable function from a chance space $(\Theta, P, \mathrm{Cr}) \times (\Omega, A, \mathrm{Pr})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{(\theta, \omega) \in \Theta \times \Omega \mid \xi(\theta, \omega) \in B\}$ is an event.

Definition B. *(Li and Liu [11])* let $(\Theta, P, \mathrm{Cr}) \times (\Omega, A, \mathrm{Pr})$ be a chance space. Then a chance measure of an event Λ is defined as

$$\mathrm{Ch}\{\Lambda\} = \begin{cases} \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}\right), \\ \qquad if \quad \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}\right) < 0.5 \\ 1 - \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda^c(\theta)\}\right), \\ \qquad if \quad \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}\right) \geq 0.5 \end{cases}$$

(2.1)

In fact, chance measure may be defined in different ways. For example, we may employ the following chance measure,

$$\mathrm{Ch}\{\Lambda\} = \frac{1}{2}\left(\sup_{\theta \in \Theta}\left(\mu(\theta) \times \mathrm{Pr}\{\Lambda(\theta)\}\right) + 1 - \sup_{\theta \in \Theta}\left(\mu(\theta) \times \mathrm{Pr}\{\Lambda^c(\theta)\}\right)\right)$$

where $\mu(\theta) = \left(2Cr\{\theta\}\right) \wedge 1$.

Theorem (a). The chance measure is self-dual. That is,

$$Ch\{\Lambda\} + Ch\{\Lambda^c\} = 1$$

For any event Λ.

Proof: For any event Λ, please note that

$$\mathrm{Ch}\{\Lambda^c\} = \begin{cases} \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda^c(\theta)\}\right), if \quad \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda^c(\theta)\}\right) < 0.5 \\ 1 - \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}\right), if \quad \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda^c(\theta)\}\right) \geq 0.5 \end{cases}$$

The argument breaks down into three cases.

Case 1: $\sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}\right) < 0.5$. For this case, we have

$$\sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda^c(\theta)\}\right) \geq 0.5$$

$$Ch\{\Lambda\} + Ch\{\Lambda^c\} = \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}\right) + 1 - \sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}\right) = 1$$

Case 2: $\sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda(\theta)\}\right) \geq 0.5$ and $\sup_{\theta \in \Theta}\left(\mathrm{Cr}\{\theta\} \wedge \mathrm{Pr}\{\Lambda^c(\theta)\}\right) < 0.5$. For this case, we have

$$Ch\{\Lambda\}+Ch\{\Lambda^c\}=1-\sup_{\theta\in\Theta}\left(\mathrm{Cr}\{\theta\}\wedge\Pr\{\Lambda^c(\theta)\}\right)+\sup_{\theta\in\Theta}\left(\mathrm{Cr}\{\theta\}\wedge\Pr\{\Lambda^c(\theta)\}\right)=1$$

Case 3: $\sup_{\theta\in\Theta}\left(\mathrm{Cr}\{\theta\}\wedge\Pr\{\Lambda(\theta)\}\right)\geq 0.5$ and

$\sup_{\theta\in\Theta}\left(\mathrm{Cr}\{\theta\}\wedge\Pr\{\Lambda^c(\theta)\}\right)\geq 0.5$

For this case,

$$\sup_{\theta\in\Theta}\left(\mathrm{Cr}\{\theta\}\wedge\Pr\{\Lambda(\theta)\}\right)=\sup_{\theta\in\Theta}\left(\mathrm{Cr}\{\theta\}\wedge\Pr\{\Lambda^c(\theta)\}\right)=0.5$$

Hence $Ch\{\Lambda\}+Ch\{\Lambda^c\}=0.5+0.5=1$. The theorem is proved.

Definition C. Let $(\Theta,P,\mathrm{Cr})\times(\Omega,A,\Pr)$ be a chance space and A , B two events. Then the conditional chance measure of A given B is defined by

$$\mathrm{Ch}\{A\mid B\}=\begin{cases}\dfrac{\mathrm{Ch}\{A\cap B\}}{\mathrm{Ch}\{B\}}, & if\ \dfrac{\mathrm{Ch}\{A\cap B\}}{\mathrm{Ch}\{B\}}<0.5\\ 1-\dfrac{\mathrm{Ch}\{A^c\cap B\}}{\mathrm{Ch}\{B\}}, & if\ 1-\dfrac{\mathrm{Ch}\{A^c\cap B\}}{\mathrm{Ch}\{B\}}<0.5\\ 0.5, & otherwise\end{cases}$$

(2.2)

provided that $Ch\{B\}>0$.

Example i. Let ξ and η be two hybrid variables. Then we have

$$\mathrm{Ch}\{\xi=x\mid\eta=y\}=\begin{cases}\dfrac{\mathrm{Ch}\{\xi=x,\eta=y\}}{\mathrm{Ch}\{\eta=y\}}, & if\ \dfrac{\mathrm{Ch}\{\xi=x,\eta=y\}}{\mathrm{Ch}\{\eta=y\}}<0.5\\ 1-\dfrac{\mathrm{Ch}\{\xi\neq x,\eta=y\}}{\mathrm{Ch}\{\eta=y\}}, & if\ 1-\dfrac{\mathrm{Ch}\{\xi\neq x,\eta=y\}}{\mathrm{Ch}\{\eta=y\}}<0.5\\ 0.5, & otherwise\end{cases}$$

(2.3)

Provided that $Ch\{\eta=y\}>0$.

Definition D. Suppose that ξ is a discrete hybrid variable taking values in $\{x_1,x_2,\cdots\}$. Then its entropy is defined by

$$H[\xi]=\sum_{i=1}^{\infty}S\left(\mathrm{Ch}\{\xi=x_i\}\right).$$

(2.4)

Where $S(t)=-t\ln t-(1-t)\ln(1-t)$. If there exists some index k such that $\mathrm{Ch}\{\xi=x_k\}=1$, and 0 otherwise, then its entropy $H[\xi]=0$. Suppose that ξ is a simple hybrid variable taking values in $\{x_1,x_2,\cdots,x_n\}$ If $\mathrm{Ch}\{\xi=x_i\}=0.5$ for all $i=1,2,\cdots,n$, then its entropy $H[\xi]=n\ln 2$. Suppose that ξ is a discrete hybrid variable taking values in $\{x_1,x_2,\cdots\}$. Then $H[\xi]\geq 0$ and equality holds if and only if ξ is essentially a deterministic/crisp number.

III. JOINT ENTROPY OF HYBRID VECTOR

In order to measure the uncertainty of hybrid vector, we give the joint entropy of two-dimensional discrete hybrid vector and two-dimensional continuous hybrid vector in this part, and study some of their properties, also show that these definitions and properties of hybrid vectors can be extended to n-dimensional.

A. The joint entropy of two-dimensional discrete hybrid vector

Definition a. Suppose that (ξ,η) is a two-dimensional discrete hybrid vector taking values (x_i,y_j) , $x_i,y_j\in\Re, i,j=1,2,\cdots$ .Then its joint entropy is defined by

$$H[\xi,\eta]=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}S\left(\mathrm{Ch}\{\xi=x_i,\eta=y_j\}\right)$$

(3.1.1)

Where $S(t)=-t\ln t-(1-t)\ln(1-t)$.

Obviously, the joint entropy of two-dimensional discrete hybrid vector only depends on the number of hybrid vector and their chance measure, rather than depend on the values of hybrid vector.

Definition b. Suppose that (ξ,η) is a two-dimensional discrete hybrid vector taking values (x_i, y_j), $x_i, y_j \in \Re, i, j = 1, 2, \cdots$, if ξ, η is independent of each other. Then its joint entropy is defined by

$$H[\xi,\eta] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} S\left(Ch\{\xi = x_i\} \wedge Ch\{\eta = y_j\}\right)$$

(3.1.2)

where $S(t) = -t\ln t - (1-t)\ln(1-t)$

Theorem (a). Suppose that (ξ,η) is a two-dimensional discrete hybrid vector taking values (x_i, y_j), $x_i, y_j \in \Re, i, j = 1, 2, \cdots$. Then $H[\xi,\eta] \ge 0$

And equality holds if and only if (ξ,η) is essentially a deterministic/crisp number.

Proof: The non-negativity is clear. In addition $H[\xi,\eta] = 0$ if and only if $Ch\{\xi = x_i, \eta = y_j\} = 0$ or 1 for each i, j. According to theorem 2.1, there exists one and only one index k and l such that $Ch\{\xi = x_k, \eta = y_l\} = 1$., for any $(i,j) \ne (k,l)$, $Ch\{\xi = x_i, \eta = y_j\} = 0$, i.e., (ξ,η) is essentially a deterministic/crisp number.

Theorem (b). Suppose that (ξ,η) is a simple hybrid vector taking values (x_i, y_j), $x_i, y_j \in \Re$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. Then $H[\xi,\eta] \le mn\ln 2$

and equality holds if and only if (ξ,η) is a two-dimensional equip-possible hybrid vector.

Proof: Since the function $S(t)$ reaches its maximum $\ln 2$ at $t = 0.5$, we have

$$H[\xi,\eta] = \sum_{i=1}^{m} \sum_{j=1}^{n} S\left(Ch\{\xi = x_i, \eta = y_j\}\right) \le mn\ln 2$$

and equality holds if and only if $Ch\{\xi = x_i, \eta = y_j\} = 0.5$ for all $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, i.e., (ξ,η) is a two-dimensional equip-possible hybrid vector.

Above theorem states that the entropy of a hybrid vector reaches its maximum when the hybrid vector is an equip-possible one. In this case, there is no preference among all the values that the hybrid vector will take.

Especially, if ξ, η is independent of each other, according to theorem 3.1.2, we have

Theorem (c). Suppose that (ξ,η) is a simple hybrid vector taking values (x_i, y_j), $x_i, y_j \in \Re$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, if ξ, η is independent of each other. Then $H[\xi,\eta] = mn\ln 2$ if and only if ξ and η are all equip-possible hybrid variable.

Proof: From definition 3.1.2 and ξ, η is independent of each other, we take that:

$$H[\xi,\eta] = mn\ln 2 \Leftrightarrow Ch\{\xi = x_i\} \wedge Ch\{\eta = y_j\} = 0.5, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n.$$

That is

$$Ch\{\xi = x_i\} = Ch\{\eta = y_j\} = 0.5, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n.$$

So, ξ and η are all equip-possible hybrid variable.

B. The joint entropy of two-dimensional continuous hybrid vector

Definition a. Suppose that (ξ,η) is a two-dimensional continuous hybrid vector. Then its joint entropy is defined by

$$H[\xi,\eta] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S\left(Ch\{\xi = x, \eta = y\}\right) dxdy$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

Definition b. Suppose that (ξ,η) is a two-dimensional continuous hybrid vector, if ξ, η is independent of each other. Then its joint entropy is defined by

$$H[\xi,\eta]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}S\left(Ch\{\xi=x\}\wedge Ch\{\eta=y\}\right)dxdy$$

where $S(t)=-t\ln t-(1-t)\ln(1-t)$.

Theorem (a). Suppose that (ξ,η) is a two-dimensional continuous hybrid vector, then $H[\xi,\eta]>0$

Proof: $H[\xi,\eta]>0$ is clear. In addition, when a two-dimensional continuous hybrid vector tends to be a deterministic/crisp number, its joint entropy tends to 0. However, a deterministic/crisp number is not a two-dimensional hybrid vector.

Theorem (b). Suppose that (ξ,η) is a two-dimensional continuous hybrid vector taking values in $[a,b]\times[c,d]$, then

$$H[\xi,\eta]\le(b-a)(d-c)\ln 2$$

and equality holds if and only if (ξ,η) is a two- dimensional equip-possible hybrid vector.

Proof: Since the function $S(t)$ reaches its maximum $\ln 2$ at $t=0.5$, then, we get the conclusion.

Especially, if ξ,η is independent of each other, according to theorem 3.2.2, we have

Theorem (c). Suppose that (ξ,η) is a two-dimensional continuous hybrid vector taking values in $[a,b]\times[c,d]$,if ξ,η is independent of each other, then

$$H[\xi,\eta]=(b-a)(d-c)\ln 2$$

If and only if ξ and η are all equip-possible hybrid variable.

Proof: Since ξ,η is independent of each other and theorem 3.2.2, then we can get the above conclusion.

Theorem (d). Suppose that (ξ,η) is a two-dimensional continuous hybrid vector, for any $a\in\Re,(c,d)\in\Re^2$,then

$$H[a(\xi,\eta)+(c,d)]=a^2H[\xi,\eta]$$

Proof: According to the definition 3.2.1,we have

$$H[a(\xi,\eta)+(c,d)]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}S\left(Ch\{a\xi+c=x,a\eta+d=y\}\right)dxdy$$

$$=a^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}S\left(Ch\{\xi=u,\eta=v\}\right)dudv=a^2H[\xi,\eta]$$

C. The joint entropy of n-dimensional hybrid vector

According to the joint entropy of two-dimensional hybrid vector, we give the definition of joint entropy of n-dimensional hybrid vector in this part.

Definition a. Suppose that $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ is a n-dimensional discrete hybrid vector taking values $\left(x_1^{(j_1)},x_2^{(j_2)},\cdots,x_n^{(j_n)}\right)$, $x_i^{(j_i)}\in\Re$, $i=1,2,\cdots,n$, $j_i=1,2,\cdots$. Then its joint entropy is defined by

$$H[\xi_1,\xi_2,\cdots;\xi_n]=\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\cdots\sum_{j_n=1}^{\infty}S\left(Ch\left\{\xi_1=x_1^{(j_1)},\xi_2=x_2^{(j_2)},\cdots;\xi_n=x_n^{(j_n)}\right\}\right)$$

Where $S(t)=-t\ln t-(1-t)\ln(1-t)$.

Definition b. Suppose that $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ is a n-dimensional discrete hybrid vector taking values $\left(x_1^{(j_1)},x_2^{(j_2)},\cdots,x_n^{(j_n)}\right)$, $x_i^{(j_i)}\in\Re$, $i=1,2,\cdots,n$, $j_i=1,2,\cdots$,if $\xi_i,\ i=1,2,\cdots,n$ is independent of each other. Then its joint entropy is defined by

$$H[\xi_1,\xi_2,\cdots,\xi_n]=\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\cdots\sum_{j_n=1}^{\infty}S\left(\min_{1\le i\le n}Ch\left\{\xi_i=x_i^{(j_i)}\right\}\right)$$

where $S(t)=-t\ln t-(1-t)\ln(1-t)$.

Definition c. Suppose that $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ is a n-dimensional continuous hybrid vector. Then its joint entropy is defined by

$$H[\xi_1,\xi_2,\cdots;\xi_n]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}S\left(Ch\{\xi_1=x_1,\xi_2=x_2,\cdots;\xi_n=x_n\}\right)dx_1dx_2\cdots dx_n$$

Where $S(t)=-t\ln t-(1-t)\ln(1-t)$.

Definition d. Suppose that $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ is a n-dimensional continuous hybrid vector, if $\xi_i,\ i=1,2,\cdots,n$ is independent of each other. Then its joint entropy is defined by

$$H[\xi_1,\xi_2,\cdots,\xi_n]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}S\left(\min_{1\le i\le n}Ch\{\xi_i=x_i\}\right)dx_1dx_2\cdots dx_n$$

Where $S(t)=-t\ln t-(1-t)\ln(1-t)$.

D. C*onditional entropy of hybrid variable*

In order to measure the uncertainty of a hybrid variable given another, we give the definition of conditional entropy of discrete random fuzzy variable and discrete fuzzy random given another, we give the definition of conditional entropy of discrete random fuzzy variable and discrete fuzzy random the maximal of conditional entropy of discrete hybrid variable, and take the necessary and sufficient conditions when the conditional entropy of simple discrete hybrid variable take the maximum value.

Two instances conditional entropy of discrete hybrid variable

Definition a. Let ξ and η be two discrete random fuzzy variables taking values $\{x_1,x_2,\cdots\}$ and $\{y_1,y_2,\cdots\}$ with membership degrees and probability $\{\mu_1,\mu_2,\cdots\}\to\{p_1,p_2,\cdots\}$ and $\{\mu_1',\mu_2',\cdots\}\to\{p_1',p_2',\cdots\}$, respectively, where $\mu_1\vee\mu_2\vee\cdots=1$, $\mu_1'\vee\mu_2'\vee\cdots=1$ If $Ch\{\xi=x_i\}>0, i=1,2,\cdots$. Then the conditional entropy of η given ξ is defined by

$$H[\eta\mid\xi]=\sum_{i=1}^{\infty}\omega_i H[\eta\mid\xi=x_i]=\sum_{i=1}^{\infty}\omega_i\sum_{j=1}^{\infty}S(\mathrm{Ch}\{\eta=y_j\mid\xi=x_i\})$$

Where $S(t)=-t\ln t-(1-t)\ln(1-t)$

$$\omega_i=\begin{cases}\frac{1}{2}\left(\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k\le x_i\right)-\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k< x_i\right)+\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k\ge x_i\right)-\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k> x_i\right)\right)\\ \qquad if\ \max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k\le x_i\right)<0.5, \max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k\ge x_i\right)<0.5\\ \frac{1}{2}\left(2-2\times\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k> x_i\right)-2\times\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k< x_i\right)\right)\\ \qquad if\ \left(\frac{u_k}{2}\wedge p_k\right)\ge 0.5\ \max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k< x_i\right)<0.5, \max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k> x_i\right)<0.5\\ \frac{1}{2}\left(\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k\ge x_i\right)-\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k> x_i\right)+\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k\ge x_i\right)-\max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k> x_i\right)\right)\\ \qquad if\ \max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k< x_i\right)\ge 0.5\ \&\ \max\limits_k\left(\frac{u_k}{2}\wedge p_k\mid x_k\ge x_i\right)<0.5\\ \text{Similarly, five forms can be introduced}\end{cases}$$

It is easy to verify that all $\omega_i\ge 0$ and $\sum_{i=1}^{n}\omega_i\le 1$. If

$$\max_k\left(\frac{u_k}{2}\wedge p_k\mid x_k\le x_i\right)\vee\max_k\left(\frac{u_k}{2}\wedge p_k\mid x_k> x_i\right)\ge 0.5, i\in\{1,2,\cdots\},$$

then $\sum_{i=1}^{\infty}\omega_i=1$.

Definition b. Let ξ and η be two discrete fuzzy random variables taking values $\{x_1,x_2,\cdots\}$ and $\{y_1,y_2,\cdots\}$ with probability and membership degrees $\{p_1,p_2,\cdots\}\to\{\mu_1,\mu_2,\cdots\}$ and $\{p_1',p_2',\cdots\}\to\{\mu_1',\mu_2',\cdots\}$, respectively, where $p_1+p_2+\cdots=1$, $p_1'+p_2'+\cdots=1$. If $\mathrm{Ch}\{\xi=x_i\}>0, i=1,2,\cdots$. Then the conditional entropy of η given ξ is defined by

$$H[\eta\mid\xi]=\sum_{i=1}^{\infty}\omega_i H[\eta\mid\xi=x_i]=\sum_{i=1}^{\infty}\omega_i\sum_{j=1}^{\infty}S(\mathrm{Ch}\{\eta=y_j\mid\xi=x_i\})$$

where

$$S(t)=-t\ln t-(1-t)\ln(1-t)$$

It is easy to verify that all $\omega_i\ge 0$ and $\sum_{i=1}^{n}\omega_i\le 1$. If

$$\sup_{x_1,\cdots,x_i}\left(\left(\min_k \frac{u_k}{2} \mid x_k \le x_i\right)\wedge\sum_{j=1}^{i}p_j\right)\vee\sup_{x_{i+1},\cdots}\left(\left(\min_k \frac{u_k}{2} \mid x_i < x_k\right)\wedge\sum_{j=i+1}^{\infty}p_j\right)\ge 0.5, i\in\{1,2,\cdots\}$$

, then $\sum_{i=1}^{\infty}\omega_i = 1$.

$$\omega_i == \begin{cases} \left(\sup_{x_1,\cdots,x_i}\left(\left(\min_k \frac{u_k}{2} \mid x_k \le x_i\right)\wedge\sum_{j=1}^{i}p_j\right)-\sup_{x_1,\cdots,x_{i-1}}\left(\left(\min_k \frac{u_k}{2} \mid x_k < x_i\right)\wedge\sum_{j=1}^{i-1}p_j\right)\right) \\ \qquad if \ \sup_{x_1,\cdots,x_i}\left(\left(\min_k \frac{u_k}{2} \mid x_k \le x_i\right)\wedge\sum_{j=1}^{i}p_j\right)<0.5 \\ \left(1-\sup_{x_{i+1},\cdots}\left(\left(\min_k \frac{u_k}{2} \mid x_i < x_k\right)\wedge\sum_{j=i+1}^{\infty}p_j\right)-\sup_{x_1,\cdots,x_{i-1}}\left(\left(\min_k \frac{u_k}{2} \mid x_k < x_i\right)\wedge\sum_{j=1}^{i-1}p_j\right)\right) \\ \quad if \ \sup_{x_1,\cdots,x_i}\left(\left(\min_k \frac{u_k}{2} \mid x_k \le x_i\right)\wedge\sum_{j=1}^{i}p_j\right)\ge 0.5 \ \& \ \sup_{x_1,\cdots,x_{i-1}}\left(\left(\min_k \frac{u_k}{2} \mid x_k < x_i\right)\wedge\sum_{j=1}^{i-1}p_j\right)<0.5 \\ \left(\sup_{x_i,\cdots}\left(\left(\min_k \frac{u_k}{2} \mid x_i \le x_k\right)\wedge\sum_{j=i}^{\infty}p_j\right)-\sup_{x_{i+1},\cdots}\left(\left(\min_k \frac{u_k}{2} \mid x_i < x_k\right)\wedge\sum_{j=i+1}^{\infty}p_j\right)\right) \\ \quad if \ \sup_{x_1,\cdots,x_i}\left(\left(\min_k \frac{u_k}{2} \mid x_k \le x_i\right)\wedge\sum_{j=1}^{i}p_j\right)\ge 0.5 \ \& \ \sup_{x_1,\cdots,x_{i-1}}\left(\left(\min_k \frac{u_k}{2} \mid x_k < x_i\right)\wedge\sum_{j=1}^{i-1}p_j\right)\ge 0.5 \end{cases}$$

E. The properties of conditional entropy of discrete Hybrid variable

According to the above that two instances conditionalentropy of discrete hybrid variable, we study the minimization and the maximal of conditional entropy of discrete hybrid variable, and take the necessary and sufficient conditions when the conditional entropy of simple discrete hybrid variable take the maximum value.

Theorem (a). Suppose that ξ and η be two discrete hybrid variables taking values $\{x_1,x_2,\cdots\}$ and $\{y_1,y_2,\cdots\}$, respectively, $\mathrm{Ch}\{\xi=x_i\}>0, i=1,2,\cdots$, then $H[\eta\mid\xi]\ge 0$

Proof: From ω_i and the non-negativity of $S(t)$, we can easy get $H[\eta\mid\xi]\ge 0$.

Theorem (b). Suppose that ξ and η be two discrete hybrid variables taking values $X=\{x_1,x_2,\cdots\}$ and $Y=\{y_1,y_2,\cdots\}$, $\mathrm{Ch}\{\xi=x_i\}>0, i=1,2,\cdots$,then the necessary and sufficient conditions of $H[\eta\mid\xi]=0$ is that:

for each $\xi=x_i, i=1,2,\cdots$, no more than one $\eta=y_{j_i}$, $j_i\in\{1,2,\cdots\}$, $Ch\{\xi=x_i,\eta=y_{j_i}\}\ne 0$, and no less than one $\xi=x_k$, there exists one and only one $\eta=y_{j_k}$, $j_i\in\{1,2,\cdots\}$, such that $\mathrm{Ch}\{\xi=x_k,\eta=y_{j_k}\}=1$, i.e., ξ,η is deterministic/ crisp number.

Proof: If $H[\eta\mid\xi]=0$, then $\mathrm{Ch}\{\eta=y_j\mid\xi=x_i\}=0$ or 1, $i,j=1,2,\cdots$. The argument breaks down into three cases.

Case 1: $\mathrm{Ch}\{\eta=y_j\mid\xi=x_i\}\equiv 0$, $i,j=1,2,\cdots$. For this case, we have

$$\mathrm{Ch}\{\eta=y_j,\xi=x_i\}\equiv 0 \ , \ i,j=1,2,\cdots.$$

is contradiction with $\mathrm{Ch}\{\xi\in X,\eta\in Y\}=1$.

Case2: If $\xi=x_k$, there exists $\eta=y_{j_k}, j_k\in\{1,2,\cdots\}$, $\mathrm{Ch}\{\eta=y_{j_k}\mid\xi=x_k\}=1$,

$\mathrm{Ch}\{\eta=y_j\mid\xi=x_i\}=0$, $j=1,2,\cdots,i\ne k$. For this case, according to theorem 2.1, we have $\mathrm{Ch}\{\eta=y_j\mid\xi=x_k\}=0$, for any $j\ne j_k$. In addition, we reference the definition of chance measure to have

$\mathrm{Ch}\{\eta=y_j,\xi=x_k\}=0$, $\forall j\ne j_k$.

From $\mathrm{Ch}\{\eta=y_j \mid \xi=x_i\}=0$, $j=1,2,\cdots,i\neq k$, we have

$\mathrm{Ch}\{\eta=y_j,\xi=x_i\}=0, j=1,2,\cdots,i\neq k$.

So $\mathrm{Ch}\{\eta=y_j,\xi=x_i\}=0, \forall j\neq j_k, i\neq k$;

$\mathrm{Ch}\{\eta=y_{j_k},\xi=x_k\}=1$, i.e., ξ,η is deterministic/crisp number.

Case 3: If more than one x_i , there exists one and only one y_{j_i} $Ch\{\xi=x_i,\eta=y_{j_i}\}\neq 0$ Let $0<\mathrm{Ch}\{\eta=y_{j_i},\xi=x_i\}<1, i=1,2,\cdots$, $j_i\in\{1,2,\cdots\}$. According to

$\mathrm{Ch}\{\eta=y_j,\xi=x_i\}=0, \quad \forall j\neq j_i, i=1,2,\cdots,$

We have

$\mathrm{Ch}\{\eta=y_j \mid \xi=x_i\}=0, \quad \forall j\neq j_i, i=1,2,\cdots,$ From theorem 2.1, we have $\mathrm{Ch}\{\eta=y_{j_i} \mid \xi=x_i\}=1, i=1,2,\cdots, j_i\in\{1,2,\cdots\}$. So $H[\eta \mid \xi]=0.$

If only one $\xi=x_k$, there exists one and only one y_{j_k} , $j_k\in\{1,2,\cdots\}$, $Ch\{\xi=x_k,\eta=y_{j_k}\}\neq 0$ .We have $\mathrm{Ch}\{\xi=x_k,\eta=y_{j_k}\}=1$. So, ξ,η are deterministic/ crisp number x_k and y_{j_k}. $\mathrm{Ch}\{\eta=y_{j_k} \mid \xi=x_k\}=1-\frac{\mathrm{Ch}\{\eta\neq y_{j_k},\xi=x_k\}}{\mathrm{Ch}\{\xi=x_k\}}=1-0=1.$

$\Rightarrow H[\eta \mid \xi]=0.$

Theorem (c). Suppose that ξ and η be two simple discrete hybrid variables taking values $\{x_1,x_2,\cdots\}$ and $\{y_1,y_2,\cdots\}$, respectively, $\mathrm{Ch}\{\xi=x_i\}>0, i=1,2,\cdots$ .Then

$H[\eta \mid \xi]\leq n\ln 2$

and equality holds if and only if (ξ,η) is equip-possible hybrid vector.

Proof : Since the function $S(t)$ reaches its maximum $\ln 2$ at $t=0.5$ and $\sum_{i=1}^{m}\omega_i\leq 1$, we have

$$H[\eta \mid \xi]=\sum_{i=1}^{m}\omega_i\sum_{j=1}^{n}S\left(\mathrm{Ch}\{\eta=y_j \mid \xi=x_i\}\right)\leq n\ln 2$$

Equality holds if and only if $\mathrm{Ch}\{\eta=y_j \mid \xi=x_i\}=0.5$, for any $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

So, $\frac{\mathrm{Ch}\{\eta=y_j,\xi=x_i\}}{\mathrm{Ch}\{\xi=x_i\}}\geq 0.5$ and $\frac{\mathrm{Ch}\{\eta\neq y_j,\xi=x_i\}}{\mathrm{Ch}\{\xi=x_i\}}\geq 0.5$, i.e., for any $i=1,2,\cdots,m$, $j=1,2,\cdots,n$, $\mathrm{Ch}\{\eta=y_j,\xi=x_i\}=0.5\Rightarrow(\xi,\eta)$ is equip-possible hybrid vector.

Theorem (d). Suppose that ξ and η be two simple discrete hybrid variables taking values $\{x_1,x_2,\cdots,x_m\}$ and $\{y_1,y_2,\cdots,y_n\}$, respectively, $\mathrm{Ch}\{\xi=x_i\}>0$, $i=1,2,\cdots,m$. Then $H[\eta \mid \xi]=n\ln 2$ if and only if ξ , η is equip-possible hybrid variable.

Proof: According to theorem 4.2.3, we have $H[\eta \mid \xi]=n\ln 2$ if and only if $\mathrm{Ch}\{\eta=y_j\}\wedge\mathrm{Ch}\{\xi=x_i\}=0.5$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$ That is, $\mathrm{Ch}\{\eta=y_j\}=\mathrm{Ch}\{\xi=x_i\}=0.5$, $i=1,2,\cdots,m$. , $j=1,2,\cdots,n$. So, ξ, η is equip-possible hybrid variable.

IV. CONCLUSIONS

In this paper, we study the joint entropy of hybrid vector and the conditional entropy of a hybrid variable given another, and give the form of conditional entropy with discrete random fuzzy variable and discrete fuzzy random variable. According to two instances conditional entropy of discrete hybrid variable, we also study the minimization and the maximal of conditional entropy ofdiscrete hybrid variable, and take the necessary and sufficient conditions when the conditional entropy of simple discrete hybrid variable take the maximum value.

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