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Helping Solve Mathematics Competition Problems Using Computer Technology

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Abstract: The International Mathematical Olympiad (IMO) is the World Championship Mathematics Competition for High School students and is held annually in a different country. The competition paper consists of six problems from various areas of secondary school mathematics, broadly classifiable as geometry, number theory, algebra, and combinatorics. Finding the solutions to these problems requires contestants' exceptional mathematical ability and excellent mathematical knowledge. In order to identify mathematical talents, many countries organise annual national mathematics competitions. Some countries develop a systematic approach to nurture their talents via a series of enrichment training programs. In our research, we focus on one specific area of mathematics competition problems at a time, especially the problem challenge to computer. We regard this paper as the first introductory paper for our further 'Human vs Computer' research into assisting solving mathematics competition problems using computer technology. In this paper, we concentrate on 'find last digits of a number' as part of number theory. First a brief introduction of IMO is presented. Second the normal human solutions to the 'last digits of a number' are given. Then precision in Computer Science is explained in term of data types. Computer algorithm in pseudo code is present and function called FindLastDidgits is then developed in C#. Finally we develop Windows and Web applications using Microsoft Visual Studio to demonstrate that with assistance of Computer Technology, we can easily find last digits of an extremely large number with exponentiation. The developed generic utility can be used to validate the human solution in a systematic way.

Keywords: International Mathematical Olympiad (IMO); Mathematics Competition, Number Theory; Last Digits of a Number; Computer Technology; Precision; Programming; Windows Application; Web Interface.

I. INTRODUCTION

The International Mathematical Olympiad (IMO) is the World Championship Mathematics Competition for high school students and is held annually in a different country [1]. It is the oldest International Science Olympiads [2, 3]. The first IMO was held in 1959 in Romania, with 7 countries participating. It has gradually expanded to over 100 countries from five continents. The competition paper consists of six problems, with each problem being worth seven points, the total score thus being 42 points. No calculators are allowed [3]. The examination is held over two consecutive days; the contestants have four-and-a-half hours to solve three problems each day. The IMO Advisory Board ensures that the competition takes place each year and that each host country observes the regulations and traditions of the IMO [1]. The problems chosen are from various areas of secondary school mathematics, broadly classifiable as geometry, number theory, algebra, and combinatorics [1, 3]:

A. Geometry, including:

a. Properties of the orthocentre, Euler's line, nine-point-circle, Simson line, Ptolemy's inequality, Ceva and Menelaus etc.

B. Number Theory, including:

a. Fundamental Theorems on Arithmetic
b. Linear and quadratic Diophantine equations, including Pell's equation
c. Arithmetic of residues modulo n, Fermat's and Euler's theorems

C. Algebra, including:

a. Fundamental Theorems on Algebra, e.g. inequalities, factorization of a polynomial into a product of irreducible polynomials
b. Symmetric polynomials of several variables, Vieta's theorem

D. Combinatorics, including Graph theory

Finding the solutions to these problems, however, requires exceptional mathematical ability and excellent mathematical knowledge [4]. Each country has typically selection processes involving several rounds of competitions, after which the number of candidates is repeatedly reduced until the final 6 are chosen [5] to represent the country.

II. NUMBER THEORY

Number Theory is one of the most ancient and active branches of pure mathematics [6, 7]. It is mainly concerned with the properties of integers and rational numbers. In recent decades, number theoretic methods are also being used in several areas of applied mathematics, such as cryptography and coding theory. In this paper, we focus on the 'Last Digits of a Number' in Number Theory. We use the problem

Find the last three digits of 7^{9999}.

as a sample problem proposed by Victor Thebault [4].

In the following sections, we go through the last digit, the last two digits and finally the last three digits of 7^{9999}.

A. Last Digit of a Number:

Problem. Find the last digit of the number 7^{9999}.
Solution [8]:
Take consecutive powers of 7 and note the last digits.

Powers of 7	Number	Last digit
7^1	7	7
7^2	49	9
7^3	343	3
7^4	2401	1
7^5	16807	7... and it starts over
7^6	117649	9
7^7	823543	3
7^8	5764801	1

The last digits have a four-step cycle: 7, 9, 3, 1.

Since $9999 = 4 \times 2499 + 3$,

then $7^{9999} = (7^{9996})(7^3)$

Since 7^3 ends in 3,

then $7^{9999} \rightarrow (7)^3 = 3$

So the last digit is 3.

B. Last Two Digits of a Number:

Problem. Find the last two digits of the number 7^{9999}.

Solution [9]: Take a look at a few powers of 7:

Powers of 7	Number	Last two digits
7^1	7	07
7^2	49	49
7^3	343	43
7^4	2401	01
7^5	16807	07... and it starts over
7^6	117649	49
7^7	823543	43
7^8	5764801	01

So the pattern also repeats in 4 step cycles. This goes through 9999\4 = 2,499 cycles and stops at the 3rd option. So that 7^{9999} have the same last two digits as 7^3 which has 43 as its last two digits.

Therefore, the last two digits of 7^{9999} are 43.

C. Last Three Digits of a Number:

We could continue to use the same method to work out the last three digits. However, there is a more systematic approach by D. P. Richardson [4].

Problem. Find the last three digits of the integer 7^{9999}.

Solution [4]: We note that $7^4 = 2401$. Therefore, we obtain

$$7^{4n} = (2401)^n = (1+2400)^n = 1 + n \cdot 2400 + \binom{n}{2} \cdot 2400^2 + \cdots.$$

In the above expression from the third term onwards, all terms end with at least four zero digits and therefore do not influence the three final digits of the number 7^{4n}, where n □/.

In order to determine the last three digits of the integer 7^{4n}, it is enough to determine the last three digits of the integer $1 + n \cdot 2400$.

However,

$$1 + n \cdot 2400 = 24n \cdot 100 + 1$$

Consider the integer *m* to be the last digit of *24n*.

Then

$$24n \cdot 100 + 1 = (\cdots m)100 + 1 = \cdots m01$$

which means that the integers *m, 0, 1* are the last three digits of the integer

$$24n \cdot 100 + 1.$$

For *n = 2499* one has *24n = 59976* which ends up with 6. Thus, the number

$$7^{4n} = 7^{9996}$$

ends up with 601.

However, $7^3 = 343$ and therefore

$$7^{9999} = 7^{9996} \cdot 7^3 = (\cdots 601)(343) = \cdots 143$$

where $(\cdots 143)$ is easily derived if one multiplies the numbers $(\cdots 601)$ and $(\cdots 343)$.

Therefore, the last three digits of the integer 7^{9999} are the numbers 1, 4, 3.

III. COMPUTER PRECISION

In Section II, we have discussed how the contestants of the mathematics competition may solve the last digits problem without a calculator. Now we look at how we can solve the problem with the aid of a calculator or computer. Certainly it seems that the quickest way is to use a calculator to work out the value of 7^{9999}. Unfortunately the value is too large as a calculator shows *infinite* as a result [10]. Then we decide to pursue the matter with a computer programming as we want to get a precise result.

In Computer Science, we must consider the precision of a numerical quantity which is a measure of the detail in quantity. It is related to precision in mathematics, which describes the number of digits that are used to express a value [11]. Computers store numbers not with infinite precision but rather in some approximation that can be packed into a fixed number of bits (binary digits) or bytes (groups of 8 bits) [12]. Almost all computers allow the programmer a choice among several different such representations or data types. Data types can differ in the number of bits utilized (the wordlength), but also in the more fundamental aspect of whether the stored number is represented in fixed-point (int or long) or floating-point (float or double) format.

Table: 1 The Size and Range of C# Integral Types [14,15]

Type	*Size (in bits)*	*Range*
sbyte	8	-128 to 127
byte	8	0 to 255
short	16	-32768 to 32767
ushort	16	0 to 65535
int	32	-2,147,483,648 to 2,147,483,647
uint	32	0 to 4294967295
long	64	-9,223,372,036,854,775,808 to 9,223,372,036,854,775,807
ulong	64	0 to 18,446,744,073,709,551,615
char	16	0 to 65535

Table: 2 The Floating Point and Decimal Types with Size, precision, and Range [14,15].

Type	*Size (in bits)*	*precision*	*Range*
float	32	7 digits	1.5×10^{-45} to 3.4×10^{38}
double	64	15-16 digits	5.0×10^{-324} to 1.7×10^{308}
decimal	128	28-29 decimal places	1.0×10^{-28} to 7.9×10^{28}

A. Integral Types

In computer programming, an integral is a category of types. They are whole numbers, either signed or unsigned, and the char type. Table 1 shows the integral types, their size, and range [13, 14].

B. Floating Point and Decimal Types

A floating point type is either a float or double. They are used any time when a real number needs to be represented. Table 2 shows the floating point and decimal types, their size, precision, and range [14, 15].

IV. COMPUTER PROGRAMMING

We understand that 7^{9999} is an extremely large number, more precisely, an 8,451 digits number as shown below:

$$7^{9999} = \underbrace{13655326\cdots\cdots\cdots00857143}_{8,451}$$

Fortunately "last digits of a number" are within the range of precisions as shown in Table I and II in a programming language such as C# even though we cannot express the whole number like.

In the following section, we generalize the problem 'find last digits of 7^{9999}' into a generic algorithm for x^y where x, y are integers, then present the algorithm in pseudo code and coding in C# language and implement Windows and Web interfaces using Microsoft ASP .NET.

A. Algorithm

Problem. Find the last N digits of an integer x^y.

Algorithm in pseudo code is listed as follows:

```
//  FindLastDigits in pseudo code
GET Number AS x
GET Power AS y
GET number of last digits AS n
SET m to 10^n
SET LastDigits to one
FOR Index = 0 to y
     COMPUTE LastDigits As (LastDigits * x) modulus m
ENDFOR
SHOW LastDigits
```

Obviously, when we change value to 1, 2, and 3, we can get the last digit, the last two digits and the last three digits, respectively.

B. Programming in C#

We convert the pseudo code into a C# function called FindLastDigits so that it can be deployed anywhere in future coding. Please note that we have no intention to present the full coding here and some necessary debugging procedures are omitted.

```
//Find the last N digits of a number
int FindLastDigits(int aNumber, int aPower, int noOfLastDigits)
{
   //GET Number AS x
      int x = aNumber;
      //GET Power AS y
      int y = aPower;
      //GET number of last digits AS n
      int n = noOfLastDigits;
      //SET m to 10^n
   int m = Convert.ToInt32(Math.Pow(10, n));
      //SET LastDigits to one
   int lastDigits=1;

      //FOR Index = 0 to y
      //COMPUTE LastDigits As (LastDigits * x) modulus m
      //ENDFOR

   for (int i = 1; i <= y; i++)
   {
     lastDigits = (lastDigits * x) % m;
   }
      //SHOW LastDigits
   return lastDigits;

}
```

Figure 1. FindLastDigits function in C#

C. Windows Interface using C#

Microsoft Visual Studio is used to create a Windows application as shown in Figure 2 in coding and Figure 3 as the GUI (Graphical User Interface) to calculate the 'last digits of a number' by employing FindLastDigits function as illustrated in Fig. 1.

```
protected void cmdGetLastDigits_Click(object sender, EventArgs e)
{
   int x, y, n, myLastDigits;


   //GET BaseNumber
   x = Convert.ToInt32(aNumber.Text);
   //GET Power Index
   y = Convert.ToInt32(aPower.Text);
   //GET Number of Last Digits
   n = Convert.ToInt32(noOfLastDigits.Text);


   //CALL FindLastDigits function
   myLastDigits = FindLastDigits(x, y, n);

   //SHOW the last N digits
   LastDigits.Text = Convert.ToString(myLastDigits);

}
```

Figure 2. Windows Interface using C#

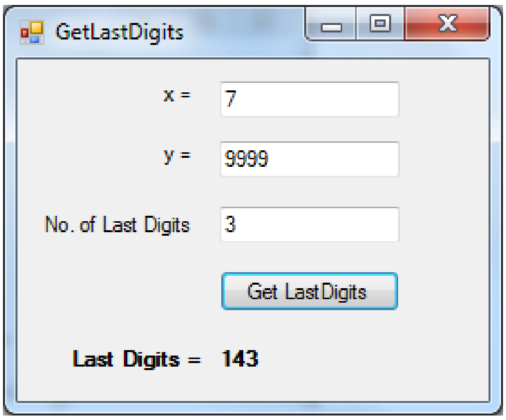

Figure 3. Windows Interface

D. Web Interface using ASP .NET

Similarly, Microsoft Visual Studio is used to create a web application to calculate the 'last digits of a number'. We easily get last 18 digits as shown in Figure 4 with some modification of data type in FindLastDigits function.

```
protected void ShowLastDigits_Click(object sender, EventArgs e)
{
   int x, y, n, myLastDigits;

   //GET BaseNumber
   x = Convert.ToInt32(aNumber.Text);
   //GET Power
   y = Convert.ToInt32(aPower.Text);
   //GET Number of Last Digits
   n = Convert.ToInt32(NoOfLastDigits.Text);


   //CALL FindLastDigits function
   myLastDigits = FindLastDigits(x, y, n);

   //SHOW the last N digits
   LastDigits.Text = Convert.ToString(myLastDigits);

}
```

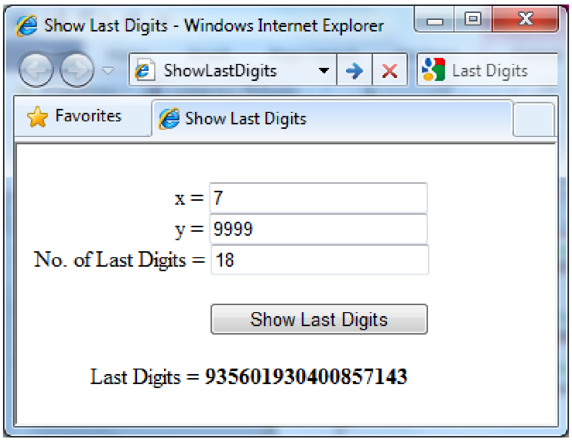

Figure 4. Web Interface

V. CONCLUSION AND FUTURE WORK

The International Mathematical Olympiad (IMO) is the international mathematics competition for high school students, which is held every year at a different country. In order to meet such a challenge, each country has rigorous selection processes which typically involve several rounds of competition, each progressively more difficult, after which the number of candidates is repeatedly reduced until the final maximum 6 are chosen [5] to represent a country. These competition problems are often to not only challenge human brain but also computer. In our first introductory paper, the research focus is on 'find last digits of a number' and use 7^{9999}, an *8,451* digits as a sample. First we generalize the problem to x^y number where x and y are integers and create FindLastDigits function. Then based on the function, we demonstrate that we can find last digits of a extremely large number using computer technology based on developed Windows and Web application using Microsoft Visual Studio 2010. Certainly it is only the tip of an iceberg and we will continue our research into more challenging mathematics competition problems using Computer Technology.

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