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# Output Regulation of the Cai Chaotic System

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*Abstract:* In this paper, we solve the problem of regulating the output of the Cai chaotic system (2007), which is one of the recently discovered three-dimensional chaotic attractors. Cai chaotic system has many interesting complex dynamical behaviours and it has potential applications in secure communication. In this paper, we construct explicit state feedback control laws to regulate the output of the Cai chaotic system so as to track constant reference signals. The control laws are derived using the regulator equations of Byrnes and Isidori (1990), who have solved the output regulation of nonlinear systems involving neutrally stable exosystem dynamics. We also discuss the simulation results in detail.

*Keywords:* Nonlinear Control Systems, Feedback Stabilization, Output Regulation, Chaos, Cai System.

## I. INTRODUCTION

Output regulation of nonlinear control systems is one of the very important problems in nonlinear control theory. The output regulation problem is the problem of controlling a fixed linear or nonlinear plant in order to have its output tracking the reference signals produced by some external generator (the exosystem). For linear control systems, the output regulation problem was solved by Francis and Wonham [1]. For nonlinear control systems, the output regulation problem was solved by Byrnes and Isidori [2] generalizing the internal model principle obtained by Francis and Wonham [1]. Byrnes and Isidori [2] made an important assumption in their work which demands that the exosystem dynamics generating the reference and disturbance signals is a neutrally stable system (Lyapunov stable in both forward and backward time). This class of exosystem signals includes the important particular cases of constant reference signals as well as sinusoidal reference signals. Using centre manifold theory for flows [3], Byrnes and Isidori derived regulator equations, which completely characterize the solution of the output regulation problem of nonlinear control systems.

The output regulation problem for linear and nonlinear control systems has been the focus of many studies in recent years ([4]-[14]). In [4], Mahmoud and Khalil obtained results on the asymptotic regulation of minimum phase nonlinear systems using output feedback. In [5], Fridman solved the output regulation problem for nonlinear control systems with delay using centre manifold theory for flows. In [6]-[7], Chen and Huang obtained results on the robust output regulation for output feedback systems with nonlinear exosystems. In [8], Liu and Huang obtained results on the global robust output regulation problem for lower triangular nonlinear systems with unknown control direction. In [9], Immonen obtained results on the practical output regulation for bounded linear infinite-dimensional state space systems. In [10], Pavlov, van de Wouw and Nijmeijer obtained results on the global nonlinear output regulation using convergence-based controller design. In [11], Xi and Ding obtained results on the global adaptive output regulation of a class of nonlinear systems with nonlinear exosystems. In [12]-[14], Serrani, Marconi and Isidori obtained results on the semi-global and global output regulation problem for minimum-phase nonlinear systems.

In this paper, we solve the output regulation problem for the Cai chaotic system (2007) using the Byrnes-Isidori regulator equations [2] to derive the state feedback control laws for regulating the output of the Cai chaotic system for the case of constant reference signals (set-point signals). The Cai chaotic system (2007) is one of the recent three-dimensional chaotic attractors studied by the scientists G. Cai and Z. Tan. Cai chaotic attractor has many interesting complex dynamical behaviours and it has potential applications in secure communication.

This paper is organized as follows. In Section II, we present a review of the solution of the output regulation for nonlinear control systems and the Byrnes-Isidori regulator equations. In Section III, we detail our solution of the output regulation problem for the Cai chaotic system. In Section IV, we discuss the simulation results. In Section V, we present the conclusions of this paper.

## II. REVIEW OF THE OUTPUT REGULATION FOR NONLINEAR CONTROL SYSTEMS

In this section, we consider a multivariable nonlinear control system modelled by equations of the form

$$\dot{x} = f(x) + g(x)u + p(x)\omega \qquad (1a)$$

$$\dot{\omega} = s(\omega) \qquad (1b)$$

$$e = h(x) - q(\omega) \qquad (2)$$

Here, the differential equation (1a) describes the plant dynamics with state $x$ defined in a neighbourhood $X$ of the origin of $\boldsymbol{R}^n$ and the input $u$ takes values in $\boldsymbol{R}^m$ subject to the effect of a disturbance represented by the vector field $p(x)\omega$. The differential equation (1b) describes an autonomous system, known as the *exosystem*, defined in a neighbourhood $W$ of the origin of $\boldsymbol{R}^k$, which models the class of disturbance and reference signals taken into consideration.

We also assume that all the constituent mappings of the system (1)-(2) and the error equation (3), namely, $f$, $g$, $p$, $s$, $h$ and $q$ are $C^1$ mappings vanishing at the origin.

Thus, for $u = 0$, the composite system (1) has an equilibrium state $(x, \omega) = (0, 0)$ with zero error (2).

A *state feedback controller* for the composite system (1) has the form

$$u = \alpha(x, \omega) \tag{3}$$

where $\alpha$ is a $C^1$ mapping defined on $X \times W$ such that $\alpha(0,0) = 0$. Upon substitution of the feedback law (3) in the composite system (1), we get the closed-loop system given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)\ \alpha(x,\omega) + p(x)\ \omega \\ \dot{\omega} &= s(\omega) \end{aligned} \tag{4}$$

The purpose of designing the state feedback controller (3) is to achieve both *internal stability* and *output regulation*. Internal stability means that when the input is disconnected from (4) [*i.e.* when $\omega = 0$], the closed-loop system (4) has an exponentially stable equilibrium at $x = 0$. Output regulation means that for the closed-loop system (4), for all initial states $(x(0), \omega(0))$ sufficiently close to the origin, $e(t) \to 0$ asymptotically as $t \to \infty$. Formally, we can summarize the requirements as follows.

**State Feedback Regulator Problem** [2]:

Find, if possible, a state feedback control law $u = \alpha(x, \omega)$ such that

(**OR1**) [**Internal Stability**] The equilibrium $x = 0$ of

$$\dot{x} = f(x) + g(x)\alpha(x, 0)$$

is locally asymptotically stable.

(**OR2**) [**Output Regulation**] There exists a neighbourhood $U$ of $(x,\omega) = (0,0)$ contained in $X \times W$ such that for each initial condition $(x(0), \omega(0))$ in $U$, the solution $(x(t), \omega(t))$ of the closed-loop system (4) satisfies

$$\lim_{t \to \infty} \left[ h(x(t)) - q(\omega(t)) \right] = 0. \qquad ■$$

Byrnes and Isidori [2] solved this problem under the following assumptions:

**(H1)** The exosystem dynamics $\dot{\omega} = s(\omega)$ is *neutrally stable* at $\omega = 0$, *i.e.* the system is Lyapunov stable in both forward and backward time at $\omega = 0$.

**(H2)** The pair $(f(x), g(x))$ has a stabilizable linear approximation at $x = 0$, *i.e.* if

$$A = \left[ \frac{\partial f}{\partial x} \right]_{x=0} \quad \text{and} \quad B = \left[ \frac{\partial g}{\partial x} \right]_{x=0},$$

then $(A, B)$ is stabilizable, which means that we can find a gain matrix $K$ so that $A + BK$ is Hurwitz. ■

Next, we recall the solution of the output regulation problem derived by Byrnes and Isidori [2].

**Theorem 1.** [2] Under the hypotheses (H1) and (H2), the state feedback regulator problem is solvable if and only if there exist $C^1$ mappings $x = \pi(\omega)$ with $\pi(0) = 0$ and $u = \varphi(\omega)$ with $\varphi(0) = 0$, both defined in a neighbourhood of $W^0 \subset W$ of $\omega = 0$ such that the following equations (called the *Byrnes-Isidori regulator equations*) are satisfied:

(1) $$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega)) + g(\pi(\omega))\varphi(\omega) + p(\pi(\omega))\omega$$

(2) $$h(\pi(\omega)) - q(\omega) = 0$$

When the Byrnes-Isidori regulator equations (1) and (2) are satisfied, a control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K\left[ x - \pi(\omega) \right] \tag{5}$$

where $K$ is any gain matrix such that $A + BK$ is Hurwitz. ■

## III. OUTPUT REGULATION OF THE CAI CHAOTIC SYSTEM

The Cai chaotic system is a new three-dimensional chaotic attractor discovered by the scientists Cai and Tan ([15], 2007) and described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= bx_1 + cx_2 - x_1 x_3 + u \\ \dot{x}_3 &= x_1^2 - hx_3 \end{aligned} \tag{6}$$

where $a > 0, b > 0, c > 0, h > 0$ are the parameters and $u$ is the control.

Cai and Tan studied the chaotic attractor (6) when the parameter values are $a = 20, b = 14, c = 10.6$ and $h = 2.8$. The chaotic portrait of the unforced Cai chaotic system is illustrated in Figure 1.

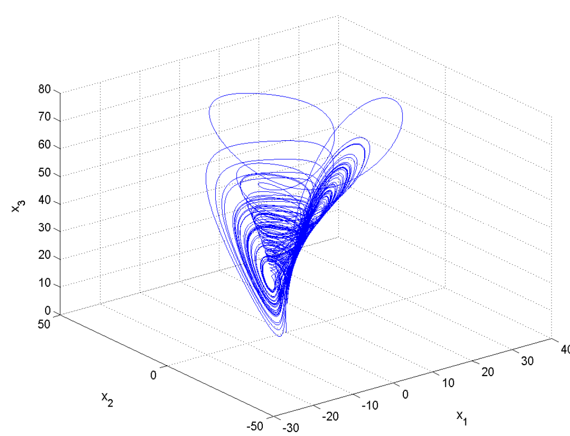


Figure 1. State Orbits of Unforced Cai Attractor (6)

In this paper, we solve the problem of output regulation for Cai chaotic attractor (6) for the tracking of constant reference signals (set-point signals).

The constant or set-point reference signals are generated by the exosystem dynamics

$$\dot{\omega} = 0 \tag{7}$$

It is important to observe that the exosystem given by (7) is neutrally stable. This follows simply because the differential equation (8) admits only constant solutions, *i.e.*

$$\omega(t) \equiv \omega(0) = \omega_0 \text{ for all } t \in \mathbf{R}. \tag{8}$$

Thus, the assumption (H1) of Theorem 1 holds trivially.

Linearizing the dynamics of the Cai chaotic attractor (6) at the equilibrium $(x_1, x_2, x_3) = (0,0,0)$, we get the following system matrices:

$$A = \begin{bmatrix} -a & a & 0 \\ b & c & 0 \\ 0 & 0 & -h \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Using Kalman's rank test for controllability [16], it can be easily seen that the pair $(A, B)$ is not controllable. However, it can be also easily seen by PBH rank test for stabilizability [16] that the pair $(A, B)$ is stabilizable. Indeed, we note that

$$\text{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = \text{rank}\begin{bmatrix} \lambda + a & -a & 0 & 0 \\ -b & \lambda - c & 0 & 1 \\ 0 & 0 & \lambda + h & 0 \end{bmatrix}$$

has rank $2$ for all values of $\lambda$ except when $\lambda = -h$. Thus, $\lambda = -h$ is an uncontrollable mode for the linear system corresponding to the system pair $(A, B)$.

Since $h > 0$, it is immediate that the uncontrollable mode $\lambda = -h$ is stable. Thus, we conclude that the pair $(A, B)$ is stabilizable. Hence, the assumption (H2) of Theorem 1 also holds.

We can also show that $(A, B)$ is stabilizable by noting that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & -h \end{bmatrix}, B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where $(A_1, B_1)$ is controllable and the uncontrollable mode $\lambda = -h$ is stable (since $h > 0$). Thus, we choose the feedback gain matrix $K$ as $K = \begin{bmatrix} \tilde{K} & 0 \end{bmatrix}$, where $\tilde{K} = \begin{bmatrix} k_1, k_2 \end{bmatrix}$ can be chosen so that the eigenvalues of $A_1 + B_1 \tilde{K}$ are arbitrarily placed in the stable region (open left-half of the complex plane).

Next, we shall discuss separately three cases of the output regulation problem detailed as follows.

**Case (A): The error equation is** $e = x_1 - \omega$

For this case, the Byrnes-Isidori regulator equations (Theorem 1) are obtained as

$$\begin{aligned} a\left[\pi_2(\omega) - \pi_1(\omega)\right] &= 0 \\ b\pi_1(\omega) + c\pi_2(\omega) - \pi_1(\omega)\pi_3(\omega) + \varphi(\omega) &= 0 \\ \pi_1^2(\omega) - h\pi_3(\omega) &= 0 \\ \pi_1(\omega) - \omega &= 0 \end{aligned} \tag{9}$$

Solving the regulator equations (9), we obtain the solution

$$\begin{aligned} &\pi_1(\omega) = \omega,\ \pi_2(\omega) = \omega,\ \pi_3(\omega) = \frac{\omega^2}{h} \\ &\varphi(\omega) = -(b+c)\omega + \frac{\omega^3}{h} \end{aligned} \tag{10}$$

By Theorem 1, the control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K[x - \pi(\omega)],$$

where $\varphi(\omega)$ and $\pi(\omega)$ are as defined in (10).

Thus, we have

$$u = -(b+c)\omega + \frac{\omega^3}{h} + k_1(x_1 - \omega) + k_2(x_2 - \omega) \tag{11}$$

**Case (B): The error equation is** $e = x_2 - \omega$

For this case, the Byrnes-Isidori regulator equations (Theorem 1) are obtained as

$$\begin{aligned} a\left[\pi_2(\omega) - \pi_1(\omega)\right] &= 0 \\ b\pi_1(\omega) + c\pi_2(\omega) - \pi_1(\omega)\pi_3(\omega) + \varphi(\omega) &= 0 \\ \pi_1^2(\omega) - h\pi_3(\omega) &= 0 \\ \pi_2(\omega) - \omega &= 0 \end{aligned} \tag{12}$$

Solving the regulator equations (9), we obtain the solution

$$\begin{aligned} &\pi_1(\omega) = \omega,\ \pi_2(\omega) = \omega,\ \pi_3(\omega) = \frac{\omega^2}{h} \\ &\varphi(\omega) = -(b+c)\omega + \frac{\omega^3}{h} \end{aligned} \tag{13}$$

By Theorem 1, the control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K[x - \pi(\omega)],$$

where $\varphi(\omega)$ and $\pi(\omega)$ are as defined in (10). Thus, we have

$$u = -(b+c)\omega + \frac{\omega^3}{h} + k_1(x_1 - \omega) + k_2(x_2 - \omega) \tag{14}$$

**Case (C): The error equation is** $e = x_3 - \omega$

For this case, the Byrnes-Isidori regulator equations (Theorem 1) are obtained as

$$\begin{aligned} a\left[\pi_2(\omega) - \pi_1(\omega)\right] &= 0 \\ b\pi_1(\omega) + c\pi_2(\omega) - \pi_1(\omega)\pi_3(\omega) + \varphi(\omega) &= 0 \\ \pi_1^2(\omega) - h\pi_3(\omega) &= 0 \\ \pi_3(\omega) - \omega &= 0 \end{aligned} \tag{15}$$

Solving the regulator equations (9), we obtain the solution

$$\begin{aligned} &\pi_1(\omega) = \sqrt{h\omega},\ \pi_2(\omega) = \sqrt{h\omega},\ \pi_3(\omega) = \omega \\ &\varphi(\omega) = \left(\omega - b - c\right)\sqrt{h\omega} \end{aligned} \tag{16}$$

By Theorem 1, the control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K[x - \pi(\omega)],$$

where $\varphi(\omega)$ and $\pi(\omega)$ are as defined in (10).

Thus, we have

$$u = (\omega - b - c)\sqrt{h\omega} + k_1(x_1 - \sqrt{h\omega}) + k_2(x_2 - \sqrt{h\omega}) \quad (17)$$

## IV. NUMERICAL SIMULATIONS

For simulation, we consider the classical chaotic case studied by Cai and Tan, *viz.* $a = 20$, $b = 14$, $c = 10.6$ and $h = 2.8$.

We also consider the set-point control as $\omega_0 = 2$.

As shown in Section III, $\lambda = -h = -2.8$ is the uncontrollable, stable eigenvalues of the closed-loop system matrix $A + BK$. Since $(A_1, B_1)$ is controllable, we can compute $\tilde{K}$ (using Ackermann's formula) so that $A_1 + B_1\tilde{K}$ has the eigenvalues $\{-4, -4\}$.

A simple calculation using MATLAB gives

$$\tilde{K} = [-26.8 \quad 1.4].$$

Thus, the gain matrix $K = [\tilde{K} \quad 0]$ is such that the closed-loop system matrix $A + BK$ has the eigenvalues

$$\{-2.8\} \cup \{-4, -4\}.$$

**Case (A): The error equation is** $e = x_1 - \omega$

Suppose that we take

$$(x_1(0), x_2(0), x_3(0)) = (5, 4, 8).$$

Also, $\omega_0 = 2$.

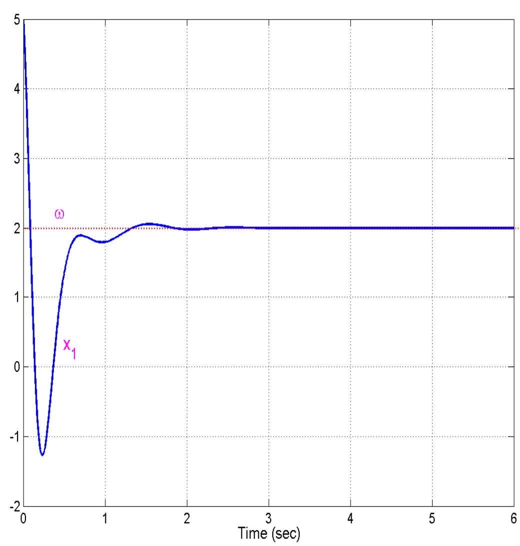


Figure 2. Case (A) - $x_1$ tracks the set-point signal $\omega$

The simulation graph is depicted in Figure 2 from which it is clear that the state $x_1(t)$ tracks the constant signal $\omega \equiv 2$ in about 4 sec.

**Case (B): The error equation is** $e = x_2 - \omega$

Suppose that we take

$$(x_1(0), x_2(0), x_3(0)) = (1, 7, 9).$$

Also, $\omega_0 = 2$.

The simulation graph is depicted in Figure 3 from which it is clear that the state $x_2(t)$ tracks the constant signal $\omega \equiv 2$ in about 4 sec.

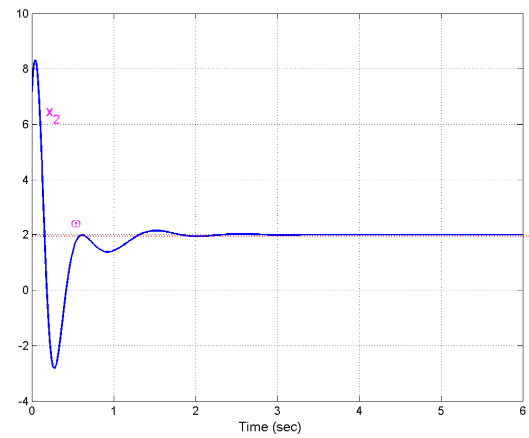


Figure 3. Case (B) - $x_2$ tracks the set-point signal $\omega$

**Case (C): The error equation is** $e = x_3 - \omega$

Suppose that we take

$$(x_1(0), x_2(0), x_3(0)) = (3, 6, 1).$$

Also, $\omega_0 = 2$.

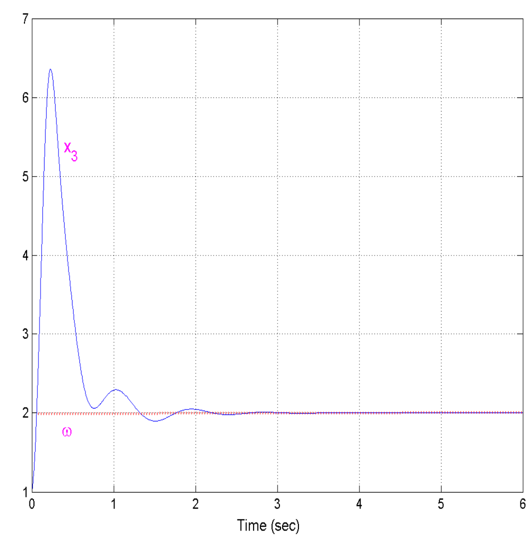


Figure 4. Case (C) - $x_3$ tracks the set-point signal $\omega$

The simulation graph is depicted in Figure 4 from which it is clear that the state $x_3(t)$ tracks the constant signal $\omega \equiv 2$ in about 4 sec.

## V. CONCLUSIONS

In this paper, we have studied in detail the output regulation of the Cai chaotic system (2007) and we have also presented the complete solution of the output regulation problem for Cai chaotic system. Explicitly, using the Byrnes-Isidori regulator equations (1990), we have presented new feedback control laws for regulating the output of the Cai chaotic system.

## VI. REFERENCES


[1] B.A. Francis and W.M. Wonham, "The internal model principle for linear multivariable regulators," Journal of Applied Mathematics and Optimization, vol. 2, pp. 170-194, 1975.

[2] C.I. Byrnes and A. Isidori, "Output regulation of nonlinear systems", IEEE Transactions on Automatic Control, vol. 35, pp. 131-140, 1990.

[3] J. Carr, Applications of Centre Manifold Theory, Springer, New York, 1981.

[4] N.A Mahmoud and H.K. Khalil, "Asymptotic regulation of minimum phase nonlinear systems using output feedback," IEEE Transactions on Automatic Control, vol. 41, pp. 1402-1412, 1996.

[5] E. Fridman, "Output regulation of nonlinear systems with delay," Systems & Control Letters, vol. 50, pp. 81-93, 2003.

[6] Z. Chen and J. Huang, "Robust output regulation with nonlinear exosystems," Automatica, vol. 41, pp. 1447-1454, 2005.

[7] Z. Chen and J. Huang, "Global robust output regulation for output feedback systems," IEEE Transactions on Automatic Control, vol. 50, pp. 117-121, 2005.

[8] L. Liu and J. Huang, "Global robust output regulation of lower triangular systems with unknown control direction," Automatica, vol. 44, pp. 1278-1284, 2008.

[9] Eero Immonen, "Practical output regulation for bounded linear infinite-dimensional state space systems," Automatica, vol. 43, 786-794, 2007.

[10] A. Pavlov, N. van de Wouw and H. Nijmeijer, "Global nonlinear output regulation: convergence-based controller design," Automatica, vol. 43, pp. 456-463, 2007.

[11] Z. Xi and Z. Ding, "Global adaptive output regulation of a class of nonlinear systems with nonlinear exosystems," Automatica, vol. 43, pp. 143-149, 2007.

[12] A. Serrani and A. Isidori, "Global robust output regulation of a class of nonlinear systems with nonlinear exosystems", Systems and Control Letters, vol. 39, pp. 133-139, 2000.

[13] A. Serrani, A. Isidori and L. Marconi, "Semiglobal output regulation for minimum phase systems," International J. Robust and Nonlinear Control, vol. 10, pp. 379-396, 2000.

[14] L. Marconi, A. Isidori and A. Serrani, "Non-resonance conditions for uniform observability in the problem of nonlinear output regulation," Systems & Control Letters, vol. 53, pp. 281-298, 2004.

[15] G. Cai and Z. Tan, "Chaos synchronization of a new chaotic system via nonlinear control," Journal of Uncertain Systems, vol. 1, pp.235-240, 2007.

[16] K. Ogata, Modern Control Engineering, Prentice Hall, New Jersey, 3rd edition, 1997.


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