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A Theory of Lattice-Valued Fuzzy Sets and Fuzzy Maps Between Different Lattice-Valued Fuzzy Sets – Revisited

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***Abstract*:** F-Set Theory is a natural generalization of Goguen's L-Fuzzy Set Theory which itself is a generalization of Zadeh's, both Fuzzy and Interval Valued Fuzzy Set Theories. It naturally and neatly extends several of the crisp (Sub)Set-Map-Properties to: L-valued f-(sub) sets, f-maps between L-valued f-sets and M-valued f-sets, where the complete lattice L-may possibly different from the complete lattice M, M-valued f-image of an L-valued f-subset of the domain L-valued f-set and L-valued f-inverse image of an M-valued f-subset of the co-domain M-valued f-set. However, for several of the results in this theory, the complete homomorphisms are assumed to be one or a combination of: 0-preserving, 0-reflecting, 1-preserving and 1-reflecting. Further, some of the results use the infinite meet distributivity of the underlying complete lattice of the domain and/or range f-set.

Now the aim of this paper is: 1. to separate this (these) hypothesis (hypotheses) of preserving/reflecting from the results in F-Set Theory and restate and prove the corresponding results and 2. to remove the hypothesis of infinite meet distributivity of the underlying complete lattice for truth values via altogether *new* proofs and 3. to add several new results that are needed/developed in this process.

Keywords: L-Fuzzy Set, L-Fuzzy Image, L-Fuzzy Inverse Image, Complete Lattice.

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I. INTRODUCTION

Zadeh introduced the notion of fuzzy subset of a set in his pioneering paper Zadeh[9] liberating mathematical logic completely from the clasps of Boolean Values taking the domain/range of applications of Mathematics to altogether new fields that were unimagined even at the times of its inception.

According to Zadeh[9], a fuzzy subset of a set X is any function f from the set X itself to the closed interval $[0,1]$ of real numbers. An element x , belonging to the set X , belongs to the fuzzy subset f with the degree of membership fx, a *real number* between 0 and 1 .

Goguen[1] generalized the Zadeh' Fuzzy Set Theory to even a higher level, introducing the notion of an L-fuzzy sub set of a set, which takes its truth values in an arbitrary but fixed complete lattice L.

According to Goguen[1], an L-fuzzy subset of a set X is any function f from the set X itself to an arbitrary but fixed complete lattice L. An element x , belonging to the set X , belongs to the fuzzy subset f with the degree of membership fx, a *lattice elemen* L.

However, still the following are some lacunae that one can easily observe with any of the above notions:

a. There is *no* such notion as fuzzy set (of course some mathematicians observed that one can define the notion of a fuzzy set to be the constant map assuming the value 1, but it was *not exploited* further.)

b. It is predominant in Mathematics that, for a pair of objects to be considered one as a sub object of the other, they both must be of the same type, namely, both objects are sets, both objects are pairs, both objects are triplets etc. and this *type compatibility* between set and its fuzzy subset is *absent* in the sense that fuzzy subset is a map while the set is *not*. (Of course, one can make here two arguments namely, a map is a particular type of relation which is a subset and hence a set, and thus a fuzzy subset is also a set and secondly one can identify a set with the map that takes the constant value 1; but both of them are *not* completely natural.)

c. There is *no* such notion as fuzzy map between fuzzy sets with truth values in *different* lattices

d. It is *not* possible to accommodate the notions of fuzzy weak-relative-sub algebra and fuzzy strong-relative-subalgebra in the *conventional* way

e. The Axiom of Choice is *not* extendable to fuzzy subsets without its dependence on the nature of the complete lattice where the fuzzy subset takes its truth values in. (Observe that the Axiom of Choice fails with the existing definitions of L-fuzzy set and L-fuzzy product as: For any pair of fuzzy sets $\overline{A},\overline{B}: X \to L$, the fuzzy product $\overline{A}\times\overline{B}$ is defined to be the fuzzy set $(\overline{A}\times\overline{B})(x) = \overline{A}x \wedge \overline{B}x$ for all $x \in X$. Letting L to be the four element diamond looking lattice with two incomparable elements α and β and letting $\overline{A}$ and $\overline{B}$ to be the constant fuzzy sets with values α and β respectively, the fuzzy product $\overline{A}\times\overline{B}$ turns out to be the empty fuzzy subset given by the constant map assuming the value 0 of L while the fuzzy subsets $\overline{A}$ and $\overline{B}$ are non-empty.

f. There is *no transparent* forgetful functor from the category of fuzzy topological spaces to the category of topological spaces which forgets the fuzzy structure.

g. There is *no transparent* forgetful functor from the category of fuzzy rings to the category of rings which forgets the fuzzy structure.

h. Last but not least, in some L-fuzzy subsets of a set, one *must* assign the value 0 for some elements of the set when actually the membership value for them is either *not* available or *not* relevant because for a fuzzy subset of a set *every* member of the set *must* be assigned a membership value.

Keeping these things in mind, Murthy[2] modified the definition of an L-fuzzy subset of a set to that of an f-set, addressing the first, second, fifth and the eighth issues above, in such a way that each f-set carries along

a) its underlying set
b) its complete lattice where the fuzzy set takes its truth values for members of its underlying set
c) its fuzzy map that specifies membership values for all elements in its underlying set and this modification resolves the above mentioned issues.

Thus an f-set is a triplet $\mathsf{A} = (A, \overline{A}, L_A)$ where

(a). A is a set, called the *underlying (crisp) set* of A

(b). L_A is a complete lattice, called the *underlying complete lattice for truth values* of elements of A

(c). $\overline{A}: A \to L_A$ is a map, called the *underlying fuzzy map* that assigns a truth value for each element of A.

In the same paper Murthy[2] also introduced the notion of an *f-map between f-sets whose underlying complete lattices for truth values are possibly, completely different*, addressing the third issue above, along with other notions like f-image of an f-subset under an f-map and f-inverse image of an f-subset under an f-map and studied the standard (lattice) algebraic properties of, all f-subsets of an f-set, all f-images of f-subsets of an f-set under an f-map and of all f-inverse images of f-subsets of an f-set under an f-map.

For a settlement of *other* issues and for elementary studies of algebraic and topological (sub) structures on f-sets, one can refer to Murthy[4,5,6] and Murthy and Yogeswara[3].

For several of the results in Murthy[2], the complete homomorphisms are assumed to be one or a combination of: 0-preserving, 0-reflecting, 1-preserving and 1-reflecting (Cf.3.3.6 and 3.3.18). Also, some of the results use the infinite meet distributivity of the underlying complete lattice of the domain and/or range f-set.

This (These) hypothesis (hypotheses) of preserving / reflecting are separated from the results of Murthy[2] and the corresponding results are *re*stated and proved in this paper. Further, in the proofs of some of the results in Murthy[2], the use of infinite meet distributivity of the underlying complete lattice for truth values is made and this is avoided via altogether *new* proofs in this paper.

This paper is a part of the Ph.D. Thesis for which the second author was awarded her doctoral degree in the month of August, 2012.

In Section-1, Introduction, the goal of this paper together with its lay out is described section wise.

In Section-2, Preliminaries, we recall some basic definitions and some algebraic properties in the theory Lattices Theory like poset, least and greatest elements of a poset, (least) upper bound, (greatest) lower bound, complete lattice, complete ideal, complete homomorphisms etc., were recalled along with some of their properties which are used later.

In Section-3, Lattice Theory for f-Set Theory, results about characterisation of complete ideals; complete ideals generated by a set and a union of sets, and relations between these complete ideals; lattice algebraic properties of complete ideals; lattice algebraic properties of supremums and infimums of images, inverse images and their combinations; and lattice algebraic properties of images and inverse images of ideals are recalled and several of them will be used in the last two sections.

In Section-4, F-Set Theory, f-set, f-subsets of an f-set; lattice algebraic properties of f-subsets of an f-set; lattice theoretic relations between (crisp) subsets of the underlying set of an f-set, Goguen-fuzzy and Zadeh-fuzzy subsets of the underlying set of the f-set and the f-subsets of the f-set; f-maps between f-sets; lattice algebraic properties of the f-images and f-inverse images of f-subsets under f-maps; and several other properties are restudied from Murthy[2].

II. PRELIMINARIES

Some basic notions in Lattice Theory like poset, least and greatest elements of a poset, (least) upper bound, (greatest) lower bound, complete lattice, complete ideal, complete homomorphisms etc., along with some of their properties are freely used and they can be glimpsed from any standard text book on Lattice Theory. However, lattice theoretic results that are used later are recalled in the next section for a ready reference.

Here onwards, for notational convenience, for all posets we always take $\le$ as the partial order in discussion. However, we use a suffix of the underlying set for the $\le$ whenever there is a possibility for confusion. Now that we agreed to take uniformly $\le$ as the symbol for all partial orders in a given discussion, we might as well drop it from the pair $(P, \le)$ and simply write only P for a poset.

We adapt a similar practice even for the operations $\wedge$, $\vee$ in additional structures on posets, like (meet/join) (complete) (semi) lattices.

Always, the empty poset is a meet (join) semi lattice and also a meet (join) complete semi lattice, a meet (join) complete semi lattice is a meet (join) semi lattice and meet (join) semi lattice is a poset.

(a) For any pair of posets P and Q and for any map $f: P \to Q$ on the underlying sets of both P and Q, f is an *order preserving* map or a *monotone* map or an *isotone*, denoted again by $f: P \to Q$ iff $a \le b$ in P implies $fa \le fb$. (b) For any pair of meet (join) complete semi lattices L and M and for any map $f: L \to M$ on the underlying sets of both L and M, f is a *meet* (*join*) *complete homomorphism* from L to M, denoted again by $f: L \to M$, iff for every *non-empty* subset A of L, $f(\wedge A) = \wedge fA$ ($f(\vee A) = \vee fA$), where fA is the image of A under f (c) For any pair of complete lattices L and M and for any map $f: L \to M$ on the underlying sets of both L and M, f is a *complete homomorphism* from L to M, denoted again by $f: L \to M$, iff it is both a meet complete and a join complete homomorphism. In other words for every non empty subset A of L, $f(\wedge A) = \wedge fA$ and $f(\vee A) = \vee fA$, where fA is the image of A under f. (d) An ordering preserving map f of posets is an *order isomorphism* iff the underlying map f is a bijection. (e) A complete homomorphism f of (Complete) (Semi) Lattices is an *isomorphism* iff the underlying map f is a bijection.

III. LATTICE THEORY FOR F-SET THEORY

In this section, results about characterization of complete ideals; complete ideals generated by a set, a union of sets and relations between these complete ideals; lattice algebraic properties of complete ideals; lattice algebraic properties of supremums and infimums of images, inverse images and their combinations; and lattice algebraic properties of images and inverse images of ideals are recalled from Murthy[7]. For counter examples with regards to the tightness of the hypotheses for various of these results, one can refer to the same paper.

A. Elementary Properties Of Lattices:

The following are some of the frequently used elementary results on complete lattices.

Theorem 1.1 In any complete lattice L, thefollowing are true forall subsets $(a_i)_{i\in I}$, $(a_j)_{j\in J}$, $(b_j)_{j\in J}$ and $(a_{i,j})_{(i,j)\in I\times J}$ of L:

a. Whenever an index set I is contained in another index set J, we have $\vee_{i\in I} a_i \le \vee_{j\in J} a_j$ and $\wedge_{j\in J} a_j \le \wedge_{i\in I} a_i$
b. $\vee_{i\in I} \vee_{j\in J} a_{i,j} = \vee_{j\in J} \vee_{i\in I} a_{i,j} = \vee_{(i,j)\in I\times J} a_{i,j}$ and $\wedge_{i\in I} \wedge_{j\in J} a_{i,j} = \wedge_{j\in J} \wedge_{i\in I} a_{i,j} = \wedge_{(i,j)\in I\times J} a_{i,j}$
c. $\wedge_{i\in I} (b \wedge a_i) = b \wedge (\wedge_{i\in I} a_i)$ and $\vee_{i\in I} (b \vee a_i) = b \vee (\vee_{i\in I} a_i)$, where $b \in L$
d. $\vee_{j\in J} (a_j \vee b_j) = (\vee_{j\in J} a_j) \vee (\vee_{j\in J} b_j)$ and $\wedge_{j\in J} (a_j \wedge b_j) = (\wedge_{j\in J} a_j) \wedge (\wedge_{j\in J} b_j)$
e. $\vee_{j\in J} (a_j \wedge b_j) \le (\vee_{j\in J} a_j) \wedge (\vee_{j\in J} b_j)$ and $\wedge_{j\in J} (a_j \vee b_j) \ge (\wedge_{j\in J} a_j) \vee (\wedge_{j\in J} b_j)$
f. $b \vee (\wedge_{i\in I} a_i) \le \wedge_{i\in I} (b \vee a_i)$ and $b \wedge (\vee_{i\in I} a_i) \ge \vee_{i\in I} (b \wedge a_i)$, where $b \in L$
g. $(\wedge_{i\in I} a_i) \wedge (\wedge_{j\in J} b_j) = \wedge_{(i,j)\in (I\times J)} (a_i \wedge b_j)$
h. $(\vee_{i\in I} a_i) \vee (\vee_{j\in J} b_j) = \vee_{(i,j)\in I\times J} (a_i \vee b_j)$
i. $\vee_{j\in J} (\wedge_{i\in I} a_{ij}) \le \wedge_{i\in I} (\vee_{j\in J} a_{ij})$.

Theorem 1.2 *In any complete lattice L, the following are true, for any family $(A_i)_{i\in I}$ of subsets of L:*

a. $\vee(\cup_{i\in I} A_i) = \vee_{i\in I} (\vee A_i)$
b. $\wedge(\cup_{i\in I} A_i) = \wedge_{i\in I} (\wedge A_i)$
c. $\vee_{i\in I} (\wedge A_i) \le \wedge(\cap_{i\in I} A_i)$; in particular, $\wedge_{i\in I} (\wedge A_i) \le \wedge(\cap_{i\in I} A_i)$
d. $\vee(\cap_{i\in I} A_i) \le \wedge_{i\in I} (\vee A_i)$. However, equality holds whenever A_i are complete ideals.

B. (Complete) Sub lattices, (Complete) Ideals:

In this section several results involving the notions of (Complete) Sub lattices, (Complete) Ideals and complete ideal generated by a subset, are recalled. Further, the collection of all complete ideals of a complete lattice is shown to be a complete lattice itself.

Let us recall that a subset S of a complete lattice L is a *complete sub lattice* of L iff it is closed under both meet and join for every non empty subset of S. A subset I of a complete lattice L is a *complete ideal* of L iff it is closed under the supremum for every non empty subset of I and closed under all the elements of L that are smaller than elements of I.

Let L be a complete sub lattice of M and $b \in L$. Then the closed interval 0, b in L, denoted by $[0,b]_L$ or simply $[0,b]$ when there is no ambiguity, is defined by $[0,b]_L = \{a \in L \mid a \le b\}$.

It is easy to see that in any complete lattice L for any $b \in L$, $[0,b]_L$ is always a complete ideal.

Later on we see that any non empty complete ideal of a complete lattice is precisely of this form.

Lemma 2.1: In any complete lattice, 1. arbitrary intersection of complete ideals is a complete ideal. Consequently 2. the intersection of all complete ideals containing a given subset is a complete ideal which is unique and smallest with respect to the containment of the given subset.

Definition 2.2: In any complete lattice L, for any given subset X, the unique smallest complete ideal containing the given sub set defined as in the above Lemma is called the complete ideal generated by X and is denoted by $(X)_L$ or simply (X) when there is no ambiguity.

Theorem 2.3: *In any complete lattice L the following are true:*

a. For any subset $\phi \ne X \subseteq L$,
   a) $(X)_L = [0, \vee X]_L$ and $\vee (X)_L = \vee X$
   b) $(X)_L = X$, whenever X itself is a complete ideal consequently $(\phi)_L = \phi$.
b. For any complete ideal $\phi \ne M$ of L, $M = [0, \vee M]_L$.
c. Non empty complete ideals are precisely of the form $[0,b]$ for some $b \in L$.

d. For any pair of non empty subsets X, Y of L, we have $\vee X = \vee Y$ iff $(X)_L = (Y)_L$.

e. For any family $(X_i)_{i \in I}$ of sub sets of L, we have $(\cup_{i \in I} X_i)_L$ is the smallest complete ideal of L containing each complete ideal $(X_i)_L$ for all $i \in I$.

In particular for any family $(I_j)_{j \in J}$ of complete ideals of L, $(\cup_{j \in J} I_j)_L$ is the smallest complete ideal of L containing each of the complete ideals I_j, $j \in J$.

f. For any pair of non empty subsets X and Y of L such that for each $x \in X$ there exists $y \in Y$ such that $x \leq y$ we have $\vee X \leq \vee Y$ and $(X)_L$ is a complete ideal of $(Y)_L$.

g. For any pair of subsets X, Y of L such that $X \subseteq Y$, we have $\vee X \leq \vee Y$ and $(X)_L$ is a complete ideal of $(Y)_L$.

h. For any subset $(a_i)_{i \in I} \subseteq L$, the following are true:

(a) $\cap_{i \in I} [0, a_i]$ = $[0, \wedge_{i \in I} a_i]$

(b) $(\cup_{i \in I} [0, a_i])_L$ = $[0, \vee_{i \in I} a_i]$ whenever I is non empty.

i. The collection of all complete ideals of the given lattice L is itself a complete lattice with the least element ϕ and the largest element L where, for any family $([0, a_i])_{i \in I}$ of complete ideals of L, $\wedge_{i \in I} [0, a_i]$ = $\cap_{i \in I} [0, a_i]$ and $\vee_{i \in I} [0, a_i]$ = $(\cup_{i \in I} [0, a_i])_L$, $(= [0, \vee_{i \in I} a_i]$ whenever I is non empty$)$.

Lemma 2.4: The following are true in any complete sub lattice N:

(a). for any complete sub lattice M of N and for any subset S of M, $(S)_M \subseteq (S)_N$. However, the equality holds whenever M is a complete ideal in N.

(b). for any pair of subsets L, M of N such that L is a complete ideal of M and M is a complete ideal of N, we have L is a complete ideal of N.

(c). for any pair of complete ideals L, M of N such that L is contained in M, we have L is a complete ideal of M.

The containment in (a) above can be strict in the above if M is *not* a complete ideal of N.

C. Complete Homomorphisms:

In this section, the generalized Lattice Theoretic results in Murthy[2], involving a. the inverse of a complete homomorphism b. the partial orders of the domain and co-domain complete lattices and c. the meet and the join of both the domain and co-domain complete lattices, for 0-preserving, 1-preserving, 0-reflecting and 1-reflecting complete homomorphisms, are recalled from Murthy[7].

Definition 3.1: Let L, M be a pair of complete lattices, Let $\psi \subseteq L \times M$ be a relation and T be a subset of L. ψ is said to be

a. $(\vee, \wedge)$ complete relation on T iff for any subset S of T, $\vee \psi(\wedge_{s \in S} s)$ = $\wedge_{s \in S} (\vee \psi s)$

b. $(\wedge, \vee)$ complete relation on T iff for any subset S of T, $\wedge \psi(\vee_{s \in S} s)$ = $\vee_{s \in S} (\wedge \psi s)$

c. $(\vee, \vee)$ complete relation on T iff for any subset S of T, $\vee \psi(\vee_{s \in S} s)$ = $\vee_{s \in S} (\vee \psi s)$

d. $(\wedge, \wedge)$ complete relation on T iff for any subset S of T, $\wedge \psi(\wedge_{s \in S} s)$ = $\wedge_{s \in S} (\wedge \psi s)$

e. $\vee$-increasing on T iff for any $a, b \in T$ such that $a \leq b$, $\vee \psi a \leq \vee \psi b$

f. $\wedge$-increasing on T iff for any $a, b \in T$ such that $a \leq b$, $\wedge \psi a \leq \wedge \psi b$

Lemma 3.2: For any complete homomorphism $\eta : L \to M$ and for any $c, d \in M$ such that $d \in \eta L$ and $c \leq d$, $\vee \eta^{-1} c \leq \vee \eta^{-1} d$.

Corollary 3.3: For any complete homomorphism $\eta : L \to M$, η^{-1} is $\vee$-increasing on ηL.

The above Lemma is *not* true whenever $d \notin \eta L$.

Lemma 3.4 : For any complete homomorphism $\eta : L \to M$ and for any $c, d \in M$ such that $c \in \eta L$ and $c \leq d$, $\wedge \eta^{-1} c \leq \wedge \eta^{-1} d$.

Corollary 3.5: For any complete homomorphism $\eta : L \to M$, η^{-1} is $\wedge$-increasing on ηL.

The above Lemma is *not* true whenever $c \notin \eta L$.

Definition 3.6: For any complete homomorphism $\eta : L \to M$, (1) η is *0-p* iff $\eta 0 = 0$ or more clearly, $\eta 0_L = 0_M$ or equivalently $0_L \in \eta^{-1} 0_M$. (2) η is *1-p* iff $\eta 1 = 1$ or more clearly, $\eta 1_L = 1_M$ or equivalently $1_L \in \eta^{-1} 1_M$.

For any map between complete lattices $\eta : L \to M$, η is (1) 0-p complete homomorphism iff $\eta(\vee S)$ = $\vee(\eta S)$ for each S such that $\phi \subseteq S \subseteq L$ (2) 1-p complete homomorphism iff $\eta(\wedge S)$ = $\wedge(\eta S)$ for each S such that $\phi \subseteq S \subseteq L$.

Lemma 3.7: For any 0-p complete homomorphism $\eta : L \to M$ and for any $\phi \subseteq D \subseteq M$, $\eta(\vee \eta^{-1} D) \leq \vee D$.

The above Lemma is *not* true whenever η is *not* 0-p.

Lemma 3.8: For any 1-p complete homomorphism $\eta : L \to M$ and for any $\phi \subseteq D \subseteq M$, $\eta(\wedge \eta^{-1} D) \geq \wedge D$

The above Lemma *is not* true whenever η is *not* 1-p.

Lemma 3.9: For any complete homomorphism

$\eta : L \to M$ and for any $\phi \neq D \subseteq \eta L$, $\eta(\vee \eta^{-1} D) = \vee D$. However, D can equal ϕ whenever η is 0-p.

In the above Lemma D can*not* equal ϕ whenever η is *not* 0-p.

Lemma 3.10: For any complete homomorphism $\eta : L \to M$ and for any $\phi \neq D \subseteq \eta L$, $\eta(\wedge \eta^{-1} D) = \wedge D$. However, D can equal ϕ whenever η is 1-p.

In the above Lemma D can*not* equal ϕ whenever η is *not* 1-p.

Corollary 3.11: For any complete homomorphism $\eta : L \to M$ the following are true:

a. whenever η is 0-p, for all $\phi \subseteq D \subseteq \eta L$, $\eta(\vee \eta^{-1} D) = \vee D$
b. whenever η is 1-p, for all $\phi \subseteq D \subseteq \eta L$, $\eta(\wedge \eta^{-1} D) = \wedge D$
c. for all $\beta \in \eta L$, (a) $\eta(\vee \eta^{-1} \beta) = \beta$ and (b) $\eta(\wedge \eta^{-1} \beta) = \beta$
d. For all $\beta \in M$, (a) $\eta(\vee \eta^{-1} \beta) \leq \beta$ whenever η is 0-p and (b) $\eta(\wedge \eta^{-1} \beta) \geq \beta$ whenever η is 1-p.

Lemma 3.12: For any 0-p complete homomorphism $\eta : L \to M$ and for any $\phi \subseteq T \subseteq \eta L$,

$\wedge \eta^{-1}(\vee_{b \in T} b) = \vee_{b \in T} (\wedge \eta^{-1} b)$.

Corollary 3.13: For any 0-p complete homomorphism $\eta : L \to M$, η^{-1} is $(\wedge, \vee)$-complete on ηL.

The above Lemma is *not* true whenever η is *not* 0-p or $T \not\subseteq \eta L$.

Corollary 3.14: For any complete homomorphism $\eta : L \to M$ and for any $\phi \neq T \subseteq M$,

$\wedge \eta^{-1}(\vee_{\beta \in T} \beta) \leq \vee_{\beta \in T} (\wedge \eta^{-1} \beta)$.

However, T can equal ϕ whenever η is 0-p.

In the above statement T can*not* equal ϕ whenever η is *not* 0-p.

Lemma 3.15: For any 1-p complete homomorphism $\eta : L \to M$ and for any $\phi \subseteq T \subseteq \eta L$,

$\vee \eta^{-1}(\wedge_{b \in T} b) = \wedge_{b \in T} (\vee \eta^{-1} b)$.

Corollary 3.16: For any 1-p complete homomorphism $\eta : L \to M$, η^{-1} is $(\vee, \wedge)$-complete on ηL.

The above Lemma is *not* true whenever η is *not* 1-p or $T \not\subseteq \eta L$.

Corollary 3.17: For any complete homomorphism $\eta : L \to M$ and for any $\phi \neq T \subseteq M$,

$\vee \eta^{-1}(\wedge_{b \in T} b) \geq \wedge_{b \in T} (\vee \eta^{-1} b)$.

However, T can equal ϕ whenever η is 1-p.

In the above statement T can*not* equal ϕ whenever η is *not* 1-p.

Definition 3.18: For any complete homomorphism $\eta : L \to M$, η is

a. 0-reflecting or simply 0-r iff $\eta a = 0$ implies $a = 0$ or equivalently $\eta^{-1} 0 \subseteq \{0\}$ (Note that $\eta^{-1} 0$ may be empty).
b. 1-reflecting or simply 1-r iff $\eta a = 1$ implies $a = 1$ or equivalently $\eta^{-1} 1 \subseteq \{1\}$ (Note that $\eta^{-1} 1$ may be empty).

Lemma 3.19: For any 0-r complete homomorphism $\eta : L \to M$ and for any $\phi \subseteq T \subseteq \eta L$,

$\vee \eta^{-1}(\vee_{b \in T} b) = \vee_{b \in T} (\vee \eta^{-1} b)$ *whenever* M *is a finite chain.*

Corollary 3.20: For any 0-r complete homomorphism $\eta : L \to M$, η^{-1} is $(\vee, \vee)$-complete on ηL, whenever M is a finite chain.

The above Lemma is *not* true whenever $T \not\subseteq \eta L$, but η is 0-r.

Also, the above Lemma is *not* true whenever $\eta : L \to M$ is *not* 0-r but $T \subseteq \eta L$.

The above Lemma is *not* true whenever $\eta : L \to M$ is 0-r but M is *not* a finite chain.

Corollary 3.21: For any complete homomorphism $\eta : L \to M$ and for any $\phi \subseteq T \subseteq M$ such that $\vee T \in \eta L$, $\vee_{b \in T} (\vee \eta^{-1} b) \leq \vee \eta^{-1}(\vee_{b \in T} b)$.

A strict inequality can hold in the above Corollary.

Lemma 3.22: For any 1-r complete homomorphism $\eta : L \to M$ and for any $\phi \subseteq T \subseteq \eta L$,

$\wedge \eta^{-1}(\wedge_{b \in T} b) = \wedge_{b \in T} (\wedge \eta^{-1} b)$ *whenever* M *is a finite chain.*

Corollary 3.23: For any 1-r complete homomorphism $\eta : L \to M$, η^{-1} is $(\wedge, \wedge)$-complete on ηL, whenever M is a finite chain.

The above is *not* true whenever $T \not\subseteq \eta L$.

The above Lemma is *not* true whenever $\eta : L \to M$ is *not* 1-r.

The above Lemma is *not* true whenever M is *not* a finite chain.

Corollary 3.24: For any complete homomorphism $\eta : L \to M$ and for any $\phi \subseteq T \subseteq M$ such that $\wedge T \in \eta L$, $\wedge \eta^{-1}(\wedge_{\beta \in T} \beta) \leq \wedge_{\beta \in T} (\wedge \eta^{-1} \beta)$.

A strict inequality can hold in the above Corollary.

D. Complete Homomorphisms and Complete Ideals:

Complete ideals of a complete lattice play a *major* role throughout the Theory of f-Sets, f-Maps, L-interval valued f-sets and interval valued f-maps between L-interval valued f-sets and M-interval valued f-sets.

In this section, results involving complete ideals, complete ideals generated by subsets, complete homomorphism, complete homomorphic images of a complete ideal, complete homomorphic images of a complete ideal generated by subsets, complete homomorphic inverse images of a complete ideal and complete homomorphic inverse images of a complete ideal generated by subsets, are recalled and all these results are used in the last two sections.

Proposition 4.1: Let $\eta : L \to M$ be a complete homomorphism. Then the following are true:

a. N is a complete sub lattice of L implies ηN is a complete sub lattice of both ηL and M.

b. N is a complete ideal of L implies ηN is a complete ideal of ηL, but *not* necessarily of M.

In (b) above ηN is *not* necessarily a complete ideal of M.

Proposition 4.2: Let η: $L \to M$ be a complete homomorphism. Then the following are true:

(a) N is a complete sub lattice of M implies η^{-1} N is a complete sub lattice of L.

(b) N is a complete ideal of M implies $\eta^{-1}N$ is a complete ideal of L.

Lemma 4.3: For any complete homomorphism $\eta : L \to M$ *and for any* $a \in L$*, the following are true:*

a. Always $\eta[0, a] \subseteq [0, \eta a]$ for all $a \in L$. However $\eta[0, a] = [0, \eta a] \cap \eta L = ([0, \eta a]_{\eta L})$.

b. However, $(\eta[0, a])_M = [0, \eta a]_M$

c. $\eta[0, a] = [0, \eta a]$ whenever η is onto.

If η is *not* onto then the conclusion (3) of the above lemma is *not* true.

Lemma 4.4: For any complete homomorphism η: $L \to M$ and for any subset X of L, we have $(\eta(X)_L)_M = (\eta X)_M$.

Corollary 4.5: For any complete homomorphism η: $L \to M$ such that η is onto, we have $\eta((X)_L) = (\eta X)_M$.

Lemma 4.6: For any complete homomorphism $\eta : L \to M$, the following are true:

a. Always $\eta^{-1}[0, b] \subseteq [0, \vee \eta^{-1}([0, b] \cap \eta L)]$ for all $b \in M$. However, Equality holds in the above, whenever η is 0-p.

b. Always $[0, \vee \eta^{-1} b] \subseteq \eta^{-1}[0, b]$ for all $b \in M$, whenever η is 0-p

c. However, $[0, \vee \eta^{-1} b] = \eta^{-1}[0, b]$ for each $b \in \eta L$, whenever η is 0-p.

The conclusion (1) of the above lemma is *not* true if η is *not* 0-p.

The conclusion (2) of the above lemma is *not* true if η is *not* 0-p.

The conclusion (3) of the above lemma is *not* true if $b \notin \eta L$ but η is 0-p.

Lemma 4.7: Let $\eta : L \to M$ be a complete homomorphism. Then

(a) $\eta^{-1}(\wedge_{i \in I} [0, b_i]) = \wedge_{i \in I} \eta^{-1}[0, b_i]$ where $b_i \in \eta L, \eta$ is 0-p and 1-p.

(b) $\vee_{j \in J} \eta^{-1}[0, b_j] \subseteq \eta^{-1}[0, \vee_{j \in J} b_j]$ whenever $b_j \in \eta L$ and η is 0-p equality holds when M is a finite chain.

Lemma 4.8: For any pair of maps $\eta, \psi : X \to M$ into a complete lattice M and for any subset A of X such that $\eta \mid A \geq \psi \mid A$, we have $\wedge \eta A \geq \wedge \psi A$ and $\vee \eta A \geq \vee \psi A$.

E. Complete Lattice Of Complete Ideals Of a Complete Lattice:

In this section relations between, modularity, distributivity and the complete infinite (meet, join) distributivity of, a. the complete lattice of complete ideals in a base complete lattice and of, b. the base complete lattice itself, are recalled.

Let us recall that a complete lattice is,

a. a *complete infinite meet distributive lattice* iff it satisfies the complete infinite meet distributive law namely, $\vee_{i \in I} (a \wedge b_i) = a \wedge \vee_{i \in I} b_i$

b. a *complete infinite join distributive lattice* iff it satisfies the complete infinite join distributive law namely, $\wedge_{i \in I} (a \vee b_i) = a \vee \wedge_{i \in I} b_i$ and

c. a *complete infinite distributive lattice* iff it is both the complete infinite meet distributive lattice and the complete infinite join distributive lattice.

Further, for any complete lattice L, the collection of complete ideals of L, is itself a complete lattice with the least element ϕ, the largest element L and the meet and joined given by: For any non empty family of $([0, a_i])_{i \in I}$ of complete ideals of L, $\wedge_{i \in I} [0, a_i] = [0, \wedge_{i \in I} a_i]$ and $\vee_{i \in I} [0, a_i] = (\cup_{i \in I} [0, a_i])_L = [0, \vee_{i \in I} a_i]$.

Definition 5.1: For any complete lattice L, the complete lattice of all complete ideals of L whose meet and join are defined as above is denoted by $CI(L)$.

Theorem 5.2: For any complete lattice L, then the following are true

a. L is complete infinite meet distributive lattice iff $CI(L)$ is so

b. L is complete infinite join distributive lattice iff $CI(L)$ is so

c. L is complete infinite distributive lattice iff $CI(L)$ is so

d. L is distributive lattice iff $CI(L)$ is so

e. L is modular lattice iff $CI(L)$ is so.

IV. F-SET THEORY

As mentioned earlier in the introduction, f-Set Theory was developed in Murthy [2] as a natural generalization of Goguen's L-Fuzzy Set Theory which itself is a generalization of Zadeh's, both Fuzzy and Interval Valued Fuzzy Set Theories.

For several of the results in this paper, the complete homomorphisms are assumed to be one or a combination of: 0-preserving, 0-reflecting, 1-preserving and 1-reflecting (Cf. 3.3.6). This (These) hypothesis (hypotheses) of preserving/reflecting are separated from the results of Murthy[2] and the corresponding results are *re*proved in *this* section.

Further, in the proofs of some of the results in this paper, the use of infinite meet distributivity of the underlying complete lattice for truth values, is avoided via altogether *new* proofs in this section.

Thus in this section, f-set, f-subsets of an f-set; lattice algebraic properties of f-sub- sets of an f-set; lattice theoretic relations between (crisp) subsets of the underlying set of an f-set, Goguen-fuzzy (and hence Zadeh-fuzzy) subsets of the underlying set of the f-set and the f-subsets of the f-set; f-maps between f-sets; lattice algebraic properties of the f-images and f-inverse images of f-subsets under f-maps; and several other properties are *re*studied.

All the results of this section are naturally and neatly extended to: L-interval valued f-(sub) sets, interval valued f-maps between L-interval valued f-sets and M -interval valued f-sets, where the complete lattice L may possibly be different from the complete lattice M, M -interval valued f-image of an L-interval valued f-subset of the domain L-interval valued f-set and L-interval valued f-inverse image of an M -interval valued f-subset of the co-domain M -interval valued f-set, in our next paper Murthy-Prasanna[ ].

A. *f-Sets and f-Subsets:*

In this section the notions of f-set, (c-total, d-total,total,strong)-f-subset, f-union and f-intersection for f-subsets of an f-set are recalled from Murthy[2].

Definition 1.1: An *f-set* is a triplet $\mathsf{A} = (A, \overline{A}, L_A)$, where A is a set, called the *underlying set of/for* A, L_A is a complete lattice, called the underlying *complete lattice of truth values of/for* A and $\overline{A}: A \to L_A$ is a map, called the *underlying fuzzy map of/for* A. In an f-set A, A, L_A and $\overline{A}$ are uniquely determined.

The f-set $(A, \overline{A}, L_A)$, where $A = \phi$, the empty set with no elements, $L_A = \phi$, the empty complete lattice with no elements and $\overline{A}$ the empty map, is called the *empty f-set* and is denoted by Φ.

For any pair of f-sets $\mathsf{A} = (A, \overline{A}, L_A)$ and $\mathsf{B} = (B, \overline{B}, L_B)$, $\mathsf{A} = \mathsf{B}$ iff $A = B$, $L_A = L_B$ and $\overline{A} = \overline{B}$.

Through out this section the letters $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{X}, \mathsf{Y}, \mathsf{Z}$ together with their suffixes always denote the f-sets, unless otherwise stated. Also, any such script P *always* denotes the triplet $(P, \overline{P}, L_P)$ where P is the underlying set for the f-set P, L_P is the underlying complete lattice of the truth values for the f-set P and $\overline{P}: P \to L_P$ is the underlying fuzzy map for the f-set P.

The letters F, G always denote the f-maps $(f, L_f), (g, L_g)$ respectively.

Definition 1.2: *Let* A, B *be a pair of f-sets.*

a. A is an *f-subset* of B iff (1) A is a subset of B (2) L_A is a complete ideal of L_B (3) $\overline{A} \le \overline{B} | A$.
b. A is a *d-total* f-subset of B iff A is an f-subset of B and $A = B$
c. A is a *c-total* f-subset of B iff A is an f-subset of B and $L_A = L_B$
d. A is a *total* f-subset of B iff A is both a c-total and a d-total f-subset of B
e. A is a *strong* f-subset of B iff A is an f-subset of B and $\overline{A} = \overline{B} | A$.

The Following are easy to see:

a) Always the f-set $\Phi = (\phi, \phi, \phi)$ is an f-subset of every f-set A.
b) $\mathsf{A} = \mathsf{B}$ iff $\mathsf{A} \subseteq \mathsf{B}$ and $\mathsf{B} \subseteq \mathsf{A}$ iff $A = B$, $L_A = L_B$ and $\overline{A} = \overline{B}$.

Definition 1.3: For any family of f-subsets $(\mathsf{A}_i)_{i \in I}$ of A,

(a). the *f-union* of $(\mathsf{A}_i)_{i \in I}$, denoted by $\cup_{i \in I} \mathsf{A}_i$, is defined by the f-set A, where

a. $A = \cup_{i \in I} A_i$ is the usual set union of the collection $(A_i)_{i \in I}$ of sets

b. $L_A = \vee_{i \in I} L_{A_i}$ where $\vee_{i \in I} L_{A_i}$ is the complete ideal generated by $\cup_{i \in I} L_{A_i}$ in L_A

c. $\overline{A}: A \to L_A$ is defined by $\overline{A}a = \vee_{i \in I_a} \overline{A_i}a$, where $I_a = \{i \in I \mid a \in A_i\}$ and

(b). the *f-intersection* of $(\mathsf{A}_i)_{i \in I}$, denoted by $\cap_{i \in I} \mathsf{A}_i$, is defined by the f-set A, where

a. $A = \cap_{i \in I} A_i$ is the usual set intersection of the collection $(A_i)_{i \in I}$ of sets

b. $L_A = \cap_{i \in I} L_{A_i}$ is the usual intersection of the complete ideals of $(L_{A_i})_{i \in I}$ in L_A

c. $\overline{A}: A \to L_A$ by $\overline{A}a = \wedge_{i \in I} \overline{A_i}a$.

B. *Algebra of f-Subsets:*

In this section some (lattice) algebraic properties of the collection ofall f-subsets of an f-set are studied. Further some lattice theoretic relations between the complete lattice of all f-subsets of an f-set and the underlying complete lattice for truth values are recalled from Murthy[2].

Proposition 2.1: The set $\mathsf{F}(\mathsf{X})$ of all f-subsets of an f-set X is a complete lattice.

Proposition 2.2: For any f-set X the following are true:

a. The complete sub lattice of all c-total, strong f-subsets of X is complete isomorphic to the complete lattice of all (crisp) subsets of X.

b. Whenever $\overline{X}$ is the constant map from X assuming the value 1 of L_X the complete sub lattice of all total f-subsets of X is complete isomorphic to the complete lattice of all L_X fuzzy subsets of X (in the sense of Goguen [5]).

C. f-Maps:

In this section the notions of, an (increasing, decreasing, preserving) f-map between an L-f-set and an M-f-set and the f-composition of such f-maps were introduced.

Definition 3.1: For any pair of f-sets A and B, the pair $\mathsf{F} = (f, L_f)$ where $f: A \to B$ is a map and $L_f: L_A \to L_B$ is a complete homomorphism, is said to be an *f-map* and is denoted by $\mathsf{F}: \mathsf{A} \to \mathsf{B}$.

Definition 3.2: For any f-map $\mathsf{F}: \mathsf{A} \to \mathsf{B}$, F is

(a) *increasing*, denoted by F_i or $(f, L_f)_i$, iff $\overline{B}f \geq L_f \overline{A}$

(b) *decreasing,* denoted by F_d or $(f, L_f)_d$, iff $\overline{B}f \leq L_f \overline{A}$

(c) *preserving*, denoted by F_p or $(f, L_f)_p$, iff $\overline{B}f = L_f \overline{A}$.

Definition 3.3: For any pair of f-maps $\mathsf{F} = (f, L_f): \mathsf{A} \to \mathsf{B}$ *and* $\mathsf{G} = (g, L_g): \mathsf{B} \to \mathsf{C}$, *the f-composition of* F *by* G, *denoted by* $\mathsf{GF}: \mathsf{A} \to \mathsf{C}$, *is defined by the f-map* $\mathsf{GF} = (gf, L_g L_f)$.

D. f-Images and f-Inverse Images under f-Maps:

In this section the notions of, the M-f-image of an L-f-subset under an f-map and the L-f-inverse image of an M-f-subset under an f-map were introduced and were shown to be well defined.

As mentioned in the beginning of this paper, for several of the results in Murthy[2], the complete homomorphisms are assumed to be one or a combination of: 0-preserving, 0-reflecting, 1-preserving and 1-reflecting (Cf.3.3.6 and 3.3.18). Also, some of the results use the infinite meet distributivity of the underlying complete lattice of the domain and/or range f-set.

Now in this section this (these) hypothesis (hypotheses) of preserving/reflecting are separated from the results in this paper and the corresponding results are *re*stated and proved here. Further, in the proofs of some of the results in the same paper, the use of infinite meet distributivity of the underlying complete lattice for truth values, is avoided via altogether *new* proofs in this paper.

Definition 4.1: Let $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ be an f-map. Then

a. For any f-subset C of A, the *f-image* of C, denoted by FC, is defined by D, where

(a) $D = fC$ (b) $L_D = (L_f L_C)_{L_B}$ (C) $\overline{D}: D \to L_D$ is given by $\overline{D}d = \overline{B}d \wedge \vee L_f \overline{C}(f^{-1}d \cap C)$ for all $d \in D$.

b. For any f-subset D of B, the *inverse f-image* of D, denoted by $\mathsf{F}^{-1}\mathsf{D}$, is defined by C, where

(a) $C = f^{-1}D$ (b) $L_C = L_f^{-1} L_D$ (c) $\overline{C}: C \to L_C$ is given by $\overline{C}c = \overline{A}c \wedge \vee L_f^{-1} \overline{D} fc$ for all $c \in C$.

The following example shows that without the term, $\overline{B}d$, the f-set D need *not* be an f-subset of B:

Example 4.2: Let $\mathsf{F}: \mathsf{A} \to \mathsf{B}, \mathsf{C} \subseteq \mathsf{A}$ be given by:

$A = \{a\} = C$, $B = \{b\}$, $\overline{A} = \{(a,1)\} = \overline{C}$, $L_A = \{0,1\} = L_B = L_C$, $\overline{B} = \{(b,0)\}$, $f: A \to B$ given by $f = \{(a,b)\}$ and $L_f = \{(0,0), (1,1)\}$. Then F is a decreasing f-map because $\overline{B}fa = 0 \leq L_f \overline{A}a = 1$.

Let D = FC. Then $D = fc = \{b\}$; $L_D = (L_f L_C)_{L_B} = \{0,1\}$ and $\overline{D}b = \vee L_f \overline{C}(f^{-1}b \cap C) = 1$, implying $\mathsf{D} = (\{b\}, \{(b,1)\}, \{0,1\})$. Clearly D is not an f-subset of B because $\overline{D}b = 1$ is not less than or equal to $\overline{B}b = 0$.

E. F-Set Theory Revisited:

In this section some standard lattice algebraic properties of the collections of, M-f-images of L-f-subsets under an f-map and the L-f-inverse images of M-f-subsets under an f-map are studied in detail.

Definition 5.1: *Let* $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ *be an f-map and let* D *be an f-subset of* B. *Then* D *is said to be an* L_f*-regular f-subset of* B *iff* $L_D \subseteq L_f L_A$.

Definition 5.2: *An f-map* $\mathsf{F} = (f, L_f)$ *is*

a. *0-preserving*, or simply *0-p* iff L_f is a 0-preserving complete homomorphism (Cf.3.3.6)
b. 1-pr*eserving* or simply *1-p* iff L_f is a 1-preserving complete homomorphism (Cf.3.3.6)
c. *0-reflecting* or simply *0-r* iff L_f is a 0-reflecting complete homomorphism (Cf.3.3.6) and
d. *1-reflec*ting or simply *1-r* iff L_f is a 1-reflecting complete homomorphism (Cf.3.3.6).

Proposition 5.3: For any f-map $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ and for any pair of f-subsets A_1 and A_2 of A such that $\mathsf{A}_1 \subseteq \mathsf{A}_2$ we have $\mathsf{F}_*\mathsf{A}_1 \subseteq \mathsf{F}_*\mathsf{A}_2$ whenever * = i or d or p.

Proof : Let $\mathsf{F}_*\mathsf{A}_1 = \mathsf{D}_1$ and $\mathsf{F}_*\mathsf{A}_2 = \mathsf{D}_2$. We show that $\mathsf{D}_1 \subseteq \mathsf{D}_2$ or (1) $D_1 \subseteq D_2$ (2) L_{D_1} is a complete ideal of L_{D_2} (3) $\overline{D}_1 \leq \overline{D}_2 \mid D_1$.

Since $\mathsf{A}_1 \subseteq \mathsf{A}_2$, we have $A_1 \subseteq A_2$, L_{A_1} is a complete ideal of L_{A_2} and $\overline{A}_1 \leq \overline{A}_2 \mid A_1$.

a. Since $A_1 \subseteq A_2$, $D_1 = fA_1 \subseteq fA_2 = D_2$.

b. First, $L_{D_1} = (L_f L_{A_1})_{L_B}$, $L_{D_2} = (L_f L_{A_2})_{L_B}$.

Next, since $L_{A_1} \subseteq L_{A_2}$, we have $L_f L_{A_1} \subseteq L_f L_{A_2} \subseteq L_B$. Therefore, by 3.2.3(7) we get $\vee L_f L_{A_1} \le \vee L_f L_{A_2}$ and $L_{D_1} = (L_f L_{A_1})_{L_B}$ is a complete ideal of $(L_f L_{A_2})_{L_B} = L_{D_2}$.

c. Let $d \in D_1$. Since $A_1 \subseteq A_2$, $f^{-1}d \cap A_1 \subseteq f^{-1}d \cap A_2$. Since $\overline{A}_1 \le \overline{A}_2 \mid A_1$, $L_f \overline{A}_1 \le L_f \overline{A}_2 \mid A_1$.

Therefore, by 3.4.8, we get that $\vee L_f \overline{A}_1(f^{-1}d \cap A_1) \le \vee L_f \overline{A}_2(f^{-1}d \cap A_1) \le \vee L_f \overline{A}_2(f^{-1}d \cap A_2)$ and hence $\overline{D}_1 d = \overline{B}d \wedge \vee L_f \overline{A}_1(f^{-1}d \cap A_1) \le \overline{B}d \cap \vee L_f \overline{A}_2(f^{-1}d \cap A_2) = \overline{D}_2 d$ or $\overline{D}_1 \le \overline{D}_2 \mid D_1$.

Lemma 5.4: For any $F: A \to B$, the set $F_r(B)$ of all L_f-regular f-subsets of B is a meet complete sub semi lattice of the complete lattice $F(B)$.

***Proof*:** (1) $B_1 \subseteq B_2$ and B_2 is L_f-regular implies B_1 is L_f-regular as follows:

B_2 is L_f-regular implies $L_{B_2} \subseteq L_f L_A$ and $B_1 \subseteq B_2$ implies L_{B_1} is a complete ideal of L_{B_2}, in particular $L_{B_1} \subseteq L_{B_2}$ and hence $L_{B_1} \subseteq L_f L_A$ or B_1 is L_f-regular.

(2) Let $B_i \in F_r(B)$ for all $i \in I$ and $B = \cap_{i \in I} B_i$. Then since $B \subseteq B_i$ and B_i is L_f-regular, by (1) above B is L_f-regular.

The following example shows that $F_r(B)$ is *not* closed under finite unions:

Example 5.5: Let $F: A \to B$ be given by: $A = (\{a\}, \{(a,1)\}, \{0, \alpha, \beta, 1 \mid 0 < \alpha, \beta < 1; \alpha \parallel \beta\})$, $B = (\{b\}, \{(b,1)\}, \{0, \alpha, \beta, \gamma, 1 \mid 0 < \alpha, \beta, \gamma < 1; \alpha \parallel \beta \parallel \gamma\})$, $f = \{(a,b)\}$, $L_f = \{(0,0), (\alpha,\alpha), (\beta,\beta), (1,1)\}$, $B_1 = (\{b\}, \{(b,\alpha)\}, \{0, \alpha \mid 0 < \alpha\})$ and $B_2 = (\{b\}, \{(b,\beta)\}, \{0, \beta \mid 0 < \beta\})$.

Now B_1 is L_f-regular because $L_{B_1} = \{0, \alpha\} \subseteq \{0, \alpha, \beta, 1\} = L_f L_A$.

B_2 is L_f-regular because $L_{B_2} = \{0, \beta\} \subseteq \{0, \alpha, \beta, 1\} = L_f L_A$.

$B = B_1 \cup B_2 = (\{b\}, \{(b,1)\}, \{0, \alpha, \beta, \gamma, 1\})$. But $L_B = \{0, \alpha, \beta, \gamma, 1\} \not\subseteq \{0, \alpha, \beta, 1\} = L_f L_A$, implying that $B_1 \cup B_2$ is *not* an L_f-regular subset of B. Therefore $F_r(B)$ is *not* closed under even finite joins.

Proposition 5.6: For any f-map $F: A \to B$ and for any pair of f-subsets B_1 and B_2 of B such that $B_1 \subseteq B_2$ and B_2 is L_f-regular, we have $F_*^{-1}B_1 \subseteq F_*^{-1}B_2$ whenever $* = $ i or d or p.

Proof **:** Let $F^{-1}B_1 = A_1$. Then $A_1 = f^{-1}B_1$, $L_{A_1} = L_f^{-1} L_{B_1}$ and $\overline{A}_1 a = \overline{A}a \wedge \vee L_f^{-1} \overline{B}_1 fa$ for all $a \in A_1$.

Let $F^{-1}B_2 = A_2$. Then $A_2 = f^{-1}B_2$, $L_{A_2} = L_f^{-1} L_{B_2}$ and $A_2 a = \overline{A}a \wedge \vee L_f^{-1} \overline{B}_2 fa$ for all $a \in A_2$.

We show that $A_1 \subseteq A_2$ or (1) $A_1 \subseteq A_2$ (2) L_{A_1} is a complete ideal of L_{A_2} (3) $\overline{A}_1 \le \overline{A}_2 \mid A_1$.

Since $B_1 \subseteq B_2$, we have $B_1 \subseteq B_2$, L_{B_1} is a complete ideal of L_{B_2} and $\overline{B}_1 \le \overline{B}_2 \mid B_1$.

a. Since $B_1 \subseteq B_2$, we have $A_1 = f^{-1}B_1 \subseteq f^{-1}B_2 = A_2$.

b. First, since $L_{B_1} \subseteq L_{B_2}$, we have $L_{A_1} = L_f^{-1} L_{B_1} \subseteq L_f^{-1} L_{B_2} = L_{A_2}$.

Since L_{A_1} is a complete ideal of L_A, L_{A_2} is a complete ideal of L_A and $L_{A_1} \subseteq L_{A_2}$, we get that L_{A_1} is a complete ideal of L_{A_2}.

c. Let $a \in A_1 = f^{-1}B_1$ be fixed. Then $fa \in B_1 \subseteq B_2$. Since $\overline{A}_1 a = \overline{A}a \wedge \vee L_f^{-1} \overline{B}_1 fa$ and $\overline{A}_2 a = \overline{A}a \wedge \vee L_f^{-1} \overline{B}_2 fa$, it is enough to show that $\vee L_f^{-1} \overline{B}_1 fa \le \vee L_f^{-1} \overline{B}_2 fa$.

Since $\overline{B}_1 \le \overline{B}_2 \mid B_1$, $\overline{B}_1 fa \le \overline{B}_2 fa$.

Since $\overline{B}_2 fa \in L_f L_A$ by L_f-regularity of B_2, by join monotonicity of L_f^{-1} as in 3.3.2, we get that $\vee L_f^{-1} \overline{B}_1 fa \le \vee L_f^{-1} \overline{B}_2 fa$.

The following example shows that the above proposition is *not* true if B_2 is *not* L_f-regular:

Example 5.7: Let $F: A \to B$ be defined by: $A = (\{a\}, \{(a,1)\}, \{0, \alpha, 1 \mid 0 < \alpha < 1\})$, $B = (\{b\}, \{(b,1)\}, \{0, \alpha, \beta, 1 \mid 0 < \alpha < \beta < 1\})$, $f = \{(a,b)\}$, $L_f = \{(0,0), (\alpha,\alpha), (1,1)\}$.

Then $1 = \overline{B}fa = L_f\overline{A}a = L_f(1) = 1$ implies $F_p : A \to B$ is preserving.

Let $B_1 = (\{b\},\{(b,\alpha)\},\{0,\alpha \mid 0<\alpha\})$ and $B_2 = (\{b\},\{(b,\beta)\},\{0,\alpha,\beta \mid 0<\alpha<\beta\})$.

Then $B_1 \subseteq B_2$ because $B_1 \subseteq B_2, L_{B_1}$ is a complete ideal of L_{B_2} and $\overline{B}_1b = \alpha < \beta = \overline{B}_2b$.

Let $A_i = F_p^{-1}B_i (i=1,2)$. Then $A_1 = \{a\} = A_2$, $L_f^{-1}L_{B_1} = \{0,\alpha\} = L_{A_1}$ is a complete ideal of $L_{A_2} = \{0,\alpha\} = L_f^{-1}L_{B_2}$.

$\overline{A}_1a = \overline{A}a \wedge \vee L_f^{-1}\overline{B}_1fa = 1\wedge\alpha = \alpha \not\leq \overline{A}_2a = \overline{A}a \wedge \vee L_f^{-1}\overline{B}_2fa = 1\wedge\vee\phi = 1\wedge 0 = 0$, implying that $A_1 \not\subseteq A_2$ or $F^{-1}B_1 \not\subseteq F^{-1}B_2$.

Proposition 5.8: For any f-map F:A→B and for any f-subset C of A, C⊆F$_*^{-1}$F$_*$C, whenever $* = i$ or p.

Proof: Let FC = D. Then $D = fC$, $L_D = (L_fL_C)_{L_B}$ and $\overline{D}d = \overline{B}d \wedge \vee L_f\overline{C}(f^{-1}d\cap C)$ for all $d\in D$.

Let $F^{-1}D$ = E. Then $E = f^{-1}D$, $L_E = L_f^{-1}L_D$ and $\overline{E}e = \overline{A}e \wedge \vee L_f^{-1}\overline{D}fe$ for all $e\in E$.

We will show that $C\subseteq E$ or (1) $C\subseteq E$ (2) L_C is a complete ideal of L_E and (3) $\overline{C} \leq \overline{E}|C$.

a. $C \subseteq f^{-1}fC = f^{-1}D = E$.

b. $L_C \subseteq L_f^{-1}L_fL_C \subseteq L_f^{-1}(L_fL_C)_{L_B} = L_f^{-1}L_D = L_E$.

Now, since both L_C and L_E are complete ideals of L_A such that $L_C\subseteq L_E$, we get that L_C is a complete ideal of L_E.

c. Let $c\in C$ be fixed. Then $\overline{E}c = \overline{A}c \wedge \vee L_f^{-1}\overline{D}fc$ where $\overline{D}fc = \overline{B}fc \wedge \vee L_f\overline{C}\ (f^{-1}fc\cap C)$ $= \overline{B}fc \wedge \vee_{a\in f^{-1}fc\cap C} L_f\overline{C}a$.

Since F is increasing, $\overline{B}fc \geq L_f\overline{A}c$. But $L_f\overline{A}c \geq L_f\overline{C}c$ because $\overline{A}|C\geq\overline{C}$ and $c\in C$.

Further, for all $a\in f^{-1}fc\cap C$, $fa = fc$ and $\overline{B}fa = \overline{B}fc$. So, $\overline{B}fc = \overline{B}fa \geq L_f\overline{A}a \geq L_f\overline{C}a$ for all $a\in f^{-1}fc\cap C$.

Therefore, $\overline{B}fc \geq \vee L_f\overline{C}(f^{-1}fc\cap C)$, implying, $\overline{D}fc = \overline{B}fc \wedge \vee L_f\overline{C}(f^{-1}fc\cap C) = \vee L_f\overline{C}(f^{-1}fc\cap C)$. But $\overline{D}fc = \vee L_f\overline{C}(f^{-1}fc\cap C) = L_f(\vee\overline{C}(f^{-1}fc\cap C))$, where the last equality is due to the facts that $f^{-1}fc\cap C \neq \phi$ and hence $\overline{C}(f^{-1}fc\cap C) \neq \phi$ and L_f is a complete homomorphism, implying that $\vee\overline{C}(f^{-1}fc\cap C) \in L_f^{-1}\overline{D}fc$.

Now, since $c\in f^{-1}fc\cap C$, from the above it follows that, $\overline{C}c \leq \vee\overline{C}(f^{-1}fc\cap C) \leq \vee L_f^{-1}\overline{D}fc$ implying $\overline{E}c = \overline{A}c\wedge\vee L_f^{-1}\overline{D}fc \geq \overline{A}c\wedge\overline{C}c = \overline{C}c$, since $\overline{A}|C\geq\overline{C}$.

The following example shows that the above proposition is *not* true for decreasing f-maps:

Example 5.9: Let $F : A\to B$ be defined by: $A = (\{a\},\{a,1\},\{0,1\mid 0<1\})$, $B = (\{b\},\{b,0\},\{0,1\mid 0<1\})$, $f = \{(a,b)\}$, $L_f = \{(0,0),(1,1)\}$ *and* $C = A$.

Then $\overline{B}fa = \overline{B}b = 0 < 1 = L_f1 = L_f\overline{A}a$ implying F is decreasing.

Let D = F_dC. Then $D = fC = \{b\}$, $L_D = (L_fL_C)_{L_B} = L_B$ and $\overline{D}b = \overline{B}b\wedge\vee L_f\overline{C}(f^{-1}b\cap C) = 0\wedge 1 = 0$.

Let $E = F_d^{-1}D$. Then $E = f^{-1}D = \{a\}$, $L_E = L_f^{-1}L_D = L_f^{-1}L_B = L_A$ and $\overline{E}a = \overline{A}a\wedge\vee L_f^{-1}\overline{D}fa = 1\wedge 0 = 0$.

Further, (a) $C = \{a\} = E$ (b) $L_C = \{0,1\} = L_E$ but (c) $\overline{C}(a) = 1 \not\leq 0 = \overline{E}(a)$, implying $\overline{C} \not\leq \overline{E}|C$ or $C \not\subseteq F_d^{-1}F_dC$.

Proposition 5.10: For any 0-p f-map $F : A\to B$ and for any f-subset C of B, we have $F_*F_*^{-1}C\subseteq C$, whenever * = d or p or i.

Proof: Let $F_*^{-1}C$ = D. Then $D = f^{-1}C$, $L_D = L_f^{-1}L_C$ and $\overline{D}a = \overline{A}a\wedge\vee L_f^{-1}\overline{C}fa$ for all $a\in D$.

Let F_*D = E. Then $E = fD$, $L_E = (L_fL_D)_{L_B}$ and $\overline{E}b = \overline{B}b\wedge L_f\overline{D}(f^{-1}b\cap D)$ for all $b\in E$.

It is enough to show that (1) $E\subseteq C$ (2) L_E is a complete ideal of L_C and (3) $\overline{E} \leq \overline{C}|E$.

a. $E = fD = ff^{-1}C \subseteq C$.

b. $L_E = (L_fL_D)_{L_B} = (L_fL_f^{-1}L_C)_{L_B} \subseteq (L_C)_{L_B} = L_C$.

Further, since both L_E and L_C are complete ideals of L_B such that $L_E\subseteq L_C$, we get that L_E is a complete ideal of L_C.

c. Let $b \in E$ be fixed. Then $\overline{E}b$ = $\overline{B}b \wedge \vee L_f \overline{D}(f^{-1}b \cap D)$, where $\overline{D}a$ = $\overline{A}a \wedge \vee L_f^{-1} \overline{C} fa$.

Now for all $a \in f^{-1}b \cap D$, $fa = b$, $a \in D$ and $L_f \overline{D}a$ = $L_f \overline{A}a \wedge L_f(\vee L_f^{-1}\overline{C} fa)$ $\leq$ $L_f \overline{A}a \wedge \overline{C} fa$ $\leq$ $\overline{C} fa$ = $\overline{C}b$ for all $a \in f^{-1}b \cap D$, where the first $\leq$ is by 3.3.11(4) and the fact that F is 0-p.

Therefore, $\vee L_f \overline{D}(f^{-1}b \cap D)$ $\leq$ $\overline{C}b$ and $\overline{E}b$ = $\overline{B}b \wedge \vee L_f \overline{D}(f^{-1}b \cap D)$ $\leq$ $\overline{B}b \wedge \overline{C}b$ $\leq$ $\overline{C}b$.

The following Example shows that if F is *not* 0-p then the above proposition need *not* be true:

Example 5.11: Let $F: A \to B$ be given by: A = $(\{a\}, \{(a,1)\}, \{0,1 \mid 0<1\})$, B = $(\{b\}, \{(b,1)\}, \{0,\alpha,1 \mid 0<\alpha<1\})$, C = $(b, \{(b,0)\}, \{0,\alpha,1 \mid 0<\alpha<1\})$, f = $\{(a,b)\}$ and L_f = $\{(0,\alpha),(1,1)\}$.

Then $\overline{B}fa$ = 1 = $L_f \overline{A}a$ implying F is preserving. If $F_p^{-1}C$ = D, then D = $f^{-1}C = \{a\}$, $L_D = L_f^{-1}L_C = L_A$ and $\overline{D}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{C} fa$ = $1 \wedge \vee \phi$ = $1 \wedge 0$ = 0.

If F_pD = E, then E = fD = $\{b\}$ = C, L_E = $(L_f L_D)_{L_B}$ = L_B = L_C and $\overline{E}b$ = $\overline{B}b \wedge \vee L_f \overline{D}(f^{-1}b \cap D)$ = $1 \wedge \alpha$ = α > 0 = $\overline{C}b$, implying $F_pF_p^{-1}C$ = E $\not\subseteq$ C.

Proposition 5.12: For any 0-p f-map $F: A \to B$ such that f, L_f are one-one and for any f-subset C of A, we have C = $F_*^{-1}F_*C$ whenever * = i or p.

Proof: Let FC = D. Then $D = fC$, L_D = $(L_f L_C)_{L_B}$ and $\overline{D}d$ = $\overline{B}d \wedge \vee L_f \overline{C}(f^{-1}d \cap C)$ for all $d \in D$.

However, since f is one-one, $\overline{D}fc$ = $\overline{B}fc \wedge \vee L_f \overline{C}(f^{-1}fc \cap C)$ = $\overline{B}fc \wedge L_f \overline{C}c$ for all $c \in C$.

Let $F^{-1}D$ = E. Then E = $f^{-1}D$, L_E = $L_f^{-1}L_D$ and $\overline{E}e$ = $\overline{A}e \wedge \vee L_f^{-1}\overline{D}fe$ for all $e \in E$.

It is enough to show that E = C or (1) E = C (2) $L_E = L_D$ (3) $\overline{E} = \overline{C}$.

a. E = $f^{-1}D$ = $f^{-1}fC$ = C where the last equality is due to the fact that f is one-one.

b. L_E = $L_f^{-1}L_D$ = $L_f^{-1}(L_f L_C)_{L_B}$. Now by 3.2.3(3), L_C = $[0,a]$ for some $a \in L_A$.

By 3.4.3(2), $(L_f L_C)_{L_B}$ = $(L_f[0,a])_{L_B}$ = $[0, L_f a]$. Therefore L_E = $L_f^{-1}(L_f L_C)_{L_B}$ = $L_f^{-1}[0, L_f a]$ = $[0, \vee L_f^{-1}L_f a]$ = $[0,a]$ = L_C, where the 4th equality follows from the fact that L_f is one-one and the 3rd equality follows from 3.4.6(3), since $L_f a \in L_f L_A$ and L_f is 0-p.

c. Let $e \in E$ be fixed. Then $\overline{D}fe$ above, together with the facts (i) $L_f \overline{C}e \in L_f L_A$ (ii) L_f^{-1} is join increasing (3.3.2) and (iii) $\overline{B}fe \wedge L_f \overline{C}e$ $\leq$ $L_f \overline{C}e$ (iv) L_f is one-one implies that $\overline{E}e$ = $\overline{A}e \wedge \vee L_f^{-1}\overline{D}fe$ = $\overline{A}e \wedge \vee L_f^{-1}(\overline{B}fe \wedge L_f \overline{C}e) \leq \overline{A}e \wedge \vee L_f^{-1}(L_f \overline{C}e)$ = $\overline{A}e \wedge \overline{C}e$ = $\overline{C}e$ because $\overline{C} \leq \overline{A} \mid C$.

On the other hand, since C $\subseteq$ $F_*^{-1}F_*C$ for * = i or p by 4.5.8, we get that $\overline{C} \leq \overline{E}$ or C = E.

The following example shows that the proposition is *not* true if, only f is one-one and *not* L_f:

Example 5.13: Let $F: A \to B$ be defined by: A = $(\{a\}, \{a,1\}, \{0,\alpha,1 \mid 0<\alpha<1\})$, B = $(\{b\}, \{b,1\}, \{0,1 \mid 0<1\})$, f = $\{(a,b)\}$, L_f = $\{(0,0),(\alpha,0),(1,1)\}$ and C = $(\{a\}$, $\{a,0\}, \{0,\alpha,1 \mid 0<\alpha<1\})$. Then f is one-one; L_f is *not* one-one because $L_f 0$ = $L_f \alpha$, but $0 \neq \alpha$ and $\overline{B}fa$ = $\overline{B}b$ = 1 = $L_f \overline{A}a$ = 1 implies F is preserving.

Let D = F_pC. Then D = fC = $\{b\}$, L_D = $(L_f L_C)_{L_B}$ = L_B and $\overline{D}b$ = $\overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C)$ = $1 \wedge 0$ = 0.

Let E = $F_p^{-1}D$. Then E = $f^{-1}D$ = $\{a\}$ = C, L_E = $L_f^{-1}L_D$ = $L_f^{-1}L_B$ = $L_f^{-1}\{0,1\}$ = L_C and $\overline{E}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ = $1 \wedge \alpha$ = α $\neq$ 0 = $\overline{C}a$, implying $F_p^{-1}F_pC$ = E $\neq$ C.

The following example shows that the above Proposition is *not* true if, only L_f is one-one but f is *not*:

Example 5.14: Let $F: A \to B$ be defined by: A = $(\{a_1, a_2\}$, $\{(a_1,1), (a_2,1)\}$, $\{0,1 \mid 0<1\})$, B = $(\{b\}$, $\{b,1\}$, $\{0,1 \mid 0<1\})$, $f = \{(a_1,b),(a_2,b)\}$, L_f = $\{(0,0), (1,1)\}$ and C = $(\{a_1\}, \{a_1,1\}, \{0,1 \mid 0<1\})$.

Then f is *not* one-one, L_f is bijective. Further, $\overline{B}fa_1$ =

$1 = L_f\,\overline{A}a_1$ and $\overline{B}fa_2 = 1 = L_f\,\overline{A}a_2$ implying F is preserving.

Let D = F_pC. Then $D = fC = \{b\}$, $L_D = (L_f L_C)_{L_B}$ $= L_B$ and $\overline{D}b = \overline{B}b \wedge \vee L_f\,\overline{C}(f^{-1}b \cap C) = 1 \wedge 1 = 1$.

Let E = F_p^{-1}D. Then $E = f^{-1}D = \{a_1, a_2\} \neq C$, implying $F_pF_p^{-1}C = E \neq C$. Note here that $L_E = L_f^{-1}L_D = L_f^{-1}L_B = L_A = L_C$ and $\overline{E}a_1 = \overline{A}a_1 \wedge \vee L_f^{-1}\overline{D}fa_1 = 1 \wedge 1 = 1 = \overline{C}a_1$.

The following example shows that the above proposition is *not* true if F is decreasing and both f and L_f are bijections:

Example 5.15: Let $F: A \to B$ be defined by, A = $(\{a\}, \{a,1\}, \{0,1 \mid 0<1\})$, B = $(\{b\}, \{b,0\}, \{0,1 \mid 0<1\})$, $f = \{(a,b)\}$, $L_f = \{(0,0), (1,1)\}$ and C = A. Then both f and L_f are bijections. Further, $\overline{B}fa = 0 \leq L_f\,\overline{A}a = 1$, implying F is decreasing.

Let D = F_dC. Then $D = fC = \{b\}$, $L_D = (L_f L_C)_{L_B}$ $= L_B$ and $\overline{D}b = \overline{B}b \wedge \vee L_f\,\overline{C}(f^{-1}b \cap C) = 0 \wedge 1 = 0$.

Let E = F_d^{-1}D. Then $E = f^{-1}D = \{a\} = C$, $L_E = L_f^{-1}L_B = L_A = L_C$ and $\overline{E}a = \overline{A}a \wedge \vee L_f^{-1}\overline{D}fa = 1 \wedge 0 = 0 \neq 1 = \overline{C}a$, implying $\overline{C} \neq \overline{E}$ or $C \neq F_d^{-1}F_dC$.

Proposition 5.16: For any f-map $F: A \to B$ such that both f and L_f are onto and for any f-subset C of B, we have $F_*F_*^{-1}C = C$, whenever * = d or p.

Proof: Let D = F^{-1}C. Then $D = f^{-1}C$, $L_D = L_f^{-1}L_C$ and $\overline{D}a = \overline{A}a \wedge \vee L_f\,\overline{C}fa$ for all $a \in D$.

Let E = FD. Then $E = fD$, $L_E = (L_f L_D)_{L_B}$ and for all $b \in E$, $\overline{E}b = \overline{B}b \wedge \vee L_f\,\overline{D}(f^{-1}b \cap D)$.

We will show that E = C or (1) $C = E$ (2) $L_E = L_C$ and (3) $\overline{C} = \overline{E}$.

a. Since f is onto, $C = ff^{-1}C = fD = E$.

b. $L_E = (L_f L_D)_{L_B} = (L_f L_f^{-1}L_C)_{L_B} = (L_C)_{L_B} = L_C$, since (i) L_f is onto and hence $L_f L_f^{-1}L_C = L_C$ and (ii) complete ideal generated by a complete ideal is itself.

c. Let $b \in E = C$ be fixed. Since F is decreasing and C ⊆ B, we have $\overline{B}f \leq L_f\,\overline{A}$ and $\overline{C} \leq \overline{B}|C$. Consequently for all $d \in f^{-1}b$, $\overline{C}fd \leq \overline{B}fd \leq L_f\,\overline{A}d$. Further, since L_f is onto, $\overline{C}fd \in L_C \subseteq L_B = L_f L_A$, by 3.3.11(3), $L_f(\vee L_f^{-1}\overline{C}fd) = \overline{C}fd$ and hence $L_f\,\overline{D}d = L_f(\overline{A}d \wedge \vee L_f^{-1}\overline{C}fd) = L_f\,\overline{A}d \wedge L_f(\vee L_f^{-1}\overline{C}fd) = L_f\,\overline{A}d \wedge \overline{C}fd = \overline{C}fd = \overline{C}b$, implying $\vee L_f\,\overline{D}(f^{-1}b \cap D) = \overline{C}b$.

Now, $\overline{E}b = \overline{B}b \wedge \vee L_f\,\overline{D}(f^{-1}b \cap D) = \overline{B}b \wedge \overline{C}b = \overline{C}b$, because $\overline{C} \leq \overline{B}|C$.

The followingexample shows that theabove proposition is *not* true if F is increasing and both f and L_f are bijections:

Example 5.17: Let $F: A \to B$ be defined by: A = $(\{a\}, \{a,0\}, \{0,1 \mid 0<1\})$, B = $(\{b\}, \{b,1\}, \{0,1 \mid 0<1\})$, $f = \{(a,b)\}$, $L_f = \{(0,0), (1,1)\}$ and C = B.

Then f is a bijection, L_f is identity and $\overline{B}fa = 1 \geq L_f\,\overline{A}a = 0$, implying F is increasing.

Let D = F_i^{-1}C. Then $D = f^{-1}C = \{a\}$, $L_D = L_f^{-1}L_C = L_A$ and $\overline{D}a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}fa = 0 \wedge 1 = 0$.

Let E = F_iD. Then $E = fD = \{b\} = C$, $L_E = (L_f L_D)_{L_B} = L_B = L_C$ and $\overline{E}b = \overline{B}b \wedge \vee L_f\,\overline{D}(f^{-1}b \cap D) = 1 \wedge 0 = 0 \neq 1 = \overline{C}b$, implying $F_iF_i^{-1}C = E \neq C$.

The following example shows that the above proposition is *not* true if, only f is onto but L_f is *not*:

Example 5.18: Let $F: A \to B$ be defined by: A = $(\{a\}, \{a,1\}, \{0,\alpha,1 \mid 0<\alpha<1\})$, B = $(\{b\}, \{b,1\}, \{0,\alpha,\beta,1 \mid 0<\alpha<\beta<1\})$, $f = \{(a,b)\}$, $L_f = \{(0,0), (\alpha,\alpha), (1,1)\}$ and $C = (\{b\}, \{b,\beta\}, L_B)$.

Then f is a bijection, L_f is *not* onto and $\overline{B}fa = 1 = L_f\,\overline{A}a$, implying F is preserving.

Let D = F_p^{-1}C. Then $D = f^{-1}C = \{a\}$, $L_D = L_f^{-1}L_C = L_f^{-1}L_B = L_A$ and $\overline{D}a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}fa = 1 \wedge \vee\phi = 1 \wedge 0 = 0$.

Let E = F_pD. Then $E = fD = \{b\} = C$, $L_E = (L_f L_D)_{L_B} = (L_f L_A)_{L_B} = L_B = L_C$ and

$\overline{E}b = \overline{B}b \wedge \vee L_f \overline{D}(f^{-1}b \cap D) = 1 \wedge 0 = 0 \neq \beta = \overline{C}b$, implying $F_pF_p^{-1}C = E \neq C$.

The following example shows that the above proposition is not true if, only L_f is onto but f is *not*:

Example 5.19: Let $F: A \to B$ be given by: $A = (\{a\},\{a,1\},\{0,1 \mid 0<1\})$, $B = (\{b_1,b_2\},\{(b_1,1),(b_2,1)\},\{0,1 \mid 0<1\})$, $f = \{(a,b_1)\}$, $L_f = \{(0,0),(1,1)\}$ and $C = B$.

Then f is *not* onto, L_f is identity and $\overline{B}fa = 1 = L_f\overline{A}a$, implying F is preserving.

Let $D = F_p^{-1}C$. Then $D = f^{-1}C = \{a\}$, $L_D = L_f^{-1}L_C = L_f^{-1}L_B = L_A$ and $\overline{D}a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}fa = 1 \wedge 1 = 1$.

Let $E = F_pD$. Then $E = fD = \{b_1\} \neq C$, implying $F_pF_p^{-1}C = E \neq C$.

Proposition 5.20: For any 0-p f-map $F: A \to B$ and for any family of f-subsets $(C_j)_{j\in J}$ of A, $F_*(\cup_{j\in J}C_j) = \cup_{j\in J}F_*C_j$ whenever * = i or d or p and L_B is a complete infinite meet distributive lattice.

Proof: Let $C = \cup_{j\in J}C_j$. Then $C = \cup_{j\in J}C_j$, $L_C = \vee_{j\in J}L_{C_j} = (\cup_{j\in J}L_{C_j})_{L_A}$ and $\overline{C}: C \to L_C$ is given by $\overline{C}a = \vee_{j\in I_a}\overline{C}_ja$, $I_a = \{j \in J \mid a \in C_j\}$ for all $a \in C$.

Let $D = FC$. Then $D = fC$, $L_D = (L_fL_C)_{L_B}$ and for all $b \in D$, $\overline{D}b = \overline{B}b \wedge \vee L_f\overline{C}(f^{-1}b \cap C)$.

Let $E_j = FC_j$. Then $E_j = fC_j$, $L_{E_j} = (L_fL_{C_j})_{L_B}$ and $\overline{E}_jb = \overline{B}b \wedge \vee L_f\overline{C}_j(f^{-1}b \cap C_j)$, for all $b \in E_j$.

Let $E = \cup_{j\in J}E_j$. Then $E = \cup_{j\in J}E_j$, $L_E = \vee_{j\in J}L_{E_j}$ and $\overline{E}b = \vee_{j\in I_b}\overline{E}_jb$, where $I_b = \{j \in J \mid b \in E_j\}$, for all $b \in E$.

Now we show that D = E or (1) $D = E$ (2) $L_D = L_E$ (3) $\overline{D} = \overline{E}$.

a. $D = fC = f(\cup_{j\in J}C_j) = \cup_{j\in J}fC_j = \cup_{j\in J}E_j = E$.

b. By 3.2.3(3), $L_{C_j} = [0,\alpha_j]$ for some $\alpha_j \in L$, for each $j \in J$.

By 3.2.3(8)(b), $L_C = \vee_{j\in J}L_{C_j} = \vee_{j\in J}[0,\alpha_j] = [0,\vee_{j\in J}\alpha_j]$.

On the other hand,by 3.4.3(2), $(L_fL_{C_j})_{L_B} = (L_f[0,\alpha_j])_{L_B} = [0,L_f\alpha_j]$ and $L_D = (L_fL_C)_{L_B} = (L_f[0,\vee_{j\in J}\alpha_j])_{L_B} = [0,L_f(\vee_{j\in J}\alpha_j)] = [0,\vee_{j\in J}L_f\alpha_j]$, where the last equality is due to the fact that L_f is 0-p (needed when $J = \phi$) and is complete homomorphism.

Again by 3.2.3(8)(b), $L_E = \vee_{j\in J}L_{E_j} = \vee_{j\in J}(L_fL_{C_j})_{L_B} = \vee_{j\in J}[0,L_f\alpha_j] = [0,\vee_{j\in J}L_f\alpha_j]$.

Clearly, $L_D = L_E$.

(3): Let $y \in fC = f(\cup_{j\in J}C_j)$, $U_x = \{j \in J \mid x \in C_j\}$ and $V_y = \{j \in J \mid y \in fC_j\}$. Then for all $x \in f^{-1}y \cap C$, $U_x \neq \phi$, $V_y \neq \phi$, $fx = y$ and $x \in C$.

Further, $\overline{D}y = \overline{B}y \wedge \vee L_f\overline{C}(f^{-1}y \cap C) = \overline{B}y \wedge \vee_{x\in f^{-1}y\cap C}L_f\overline{C}x = \overline{B}y \wedge \vee_{x\in f^{-1}y\cap C}L_f(\vee_{i\in U_x}\overline{C}_ix) = \overline{B}y \wedge \vee_{x\in f^{-1}y\cap C}\vee_{i\in U_x}L_f\overline{C}_ix$.

On the other hand, since L_B is a complete infinite meet distributive lattice,

$\overline{E}y = \vee_{j\in V_y}\overline{E}_jy = \vee_{j\in V_y}(\overline{B}y \wedge \vee_{z\in f^{-1}y\cap C_j}L_f\overline{C}_jz) = \overline{B}y \wedge \vee_{j\in V_y}\vee_{z\in f^{-1}y\cap C_j}L_f\overline{C}_jz$.

Therefore it is enough to show that $\vee_{x\in f^{-1}y\cap C}\vee_{i\in U_x}L_f\overline{C}_ix = \vee_{j\in V_y}\vee_{z\in f^{-1}y\cap C_j}L_f\overline{C}_jz$.

Let $Q = \{L_f\overline{C}_jz \mid z \in f^{-1}y \cap C_j, j \in V_y\}$ and $P = \{L_f\overline{C}_ix \mid x \in f^{-1}y \cap C, i \in U_x\}$. Then clearly, it is enough to show that $P = Q$, because $\vee P = \vee_{x\in f^{-1}y\cap C}\vee_{i\in U_x}L_f\overline{C}_ix$ and $\vee Q = \vee_{j\in V_y}\vee_{z\in f^{-1}y\cap C_j}L_f\overline{C}_jz$.

Let $\alpha \in Q$. Then $\alpha = L_f\overline{C}_jz$, $z \in f^{-1}y \cap C_j$, $j \in V_y$. Since $C_j \subseteq C$, $z \in f^{-1}y \cap C$, $j \in U_z$. Therefore $z \in f^{-1}y \cap C$, $j \in U_z$ or $\alpha = L_f\overline{C}_jz \in P$, implying $Q \subseteq P$.

Let $\beta \in P$. Then $\beta = L_f\overline{C}_ix$, $x \in f^{-1}y \cap C$, $i \in U_x$. But then $x \in f^{-1}y$ and $x \in C_i$ or $x \in f^{-1}y \cap C_i$ which implies $y = fx \in fCi$ or $i \in V_y$

which in turn implies $x \in f^{-1}y \cap C_i$, $i \in V_y$ or β = $L_f \overline{C}_i x \in Q$, implying $P \subseteq Q$.

Proposition 5.21: For any 1-p f-map $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ and for any family of f-subsets $(\mathsf{C}_j)_{j\in J}$ of A,

$\mathsf{F}_*(\cap_{j\in J} \mathsf{C}_j) \subseteq \cap_{j\in J} \mathsf{F}_* \mathsf{C}_j$, whenever * = i or d or p.

Proof: Let $\mathsf{C} = \cap_{j\in J} \mathsf{C}_j$. Then $C = \cap_{j\in J} C_j$, $L_C = \wedge_{j\in J} L_{C_j} = \cap_{j\in J} L_{C_j}$ and $\overline{C}a = \wedge_{j\in J} \overline{C}_j a$, for all $a \in A$.

Let D = FC.Then $D = fC$, $L_D = (L_f L_C)_{L_B}$ and for all $b \in B$, $\overline{D}b$ = $\overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C)$.

Let E_j = FC_j. Then E_j = fC_j, L_{E_j} = $(L_f L_{C_j})_{L_B}$ and $\overline{E}_j b$ = $\overline{B}b \wedge \vee L_f \overline{C}_j (f^{-1}b \cap C_j)$, for all $b \in E_j$.

Let E = $\cap_{j\in J} \mathsf{E}_j$. Then $E = \cap_{j\in J} E_j$, $L_E = \wedge_{j\in J} L_{E_j}$ = $\cap_{j\in J} L_{E_j}$ and $\overline{E}b = \wedge_{j\in J} \overline{E}_j b$, for all $b \in E$.

Now we show that $\mathsf{D} \subseteq \mathsf{E}$ or (1) $D \subseteq E$ (2) L_D is a complete ideal in L_E (3) $\overline{D} \le \overline{E} \mid D$.

a. D = fC = $f(\cap_{j\in J} C_j) \subseteq \cap_{j\in J} fC_j$ = $\cap_{j\in J} E_j$ = E.

b. By 3.2.3(3), $L_{C_j} = [0, \alpha_j]$, for some $\alpha_j \in L_A$ and for each $j \in J$.

So, by 3.2.3(8)(a), $L_C = \wedge_{j\in J} [0, \alpha_j] = [0, \wedge_{j\in J} \alpha_j]$.

On the other hand, by 3.4.3(2), $(L_f L_{C_j})_{L_B}$ = $(L_f [0, \alpha_j])_{L_B} = [0, L_f \alpha_j]$ and

L_D = $(L_f L_C)_{L_B}$ = $(L_f [0, \wedge_{j\in J} \alpha_j])_{L_B}$ = $[0, L_f(\wedge_{j\in J} \alpha_j)]$ = $[0, \wedge_{j\in J} L_f \alpha_j]$, where the last equality is due to the fact that L_f is 1-p (needed when $J = \phi$) and is complete homomorphism.

Now L_E = $\wedge_{j\in J} L_{E_j}$ = $\wedge_{j\in J} (L_f L_{C_j})_{L_B}$ = $\wedge_{j\in J} [0, L_f \alpha_j] = [0, \wedge_{j\in J} L_f \alpha_j]$. Therefore $L_D = L_E$.

c. Let $y \in fC = f(\cap_{j\in J} C_j)$ be fixed.

Then $\overline{D}y$ = $\overline{B}y \wedge \vee L_f \overline{C}(f^{-1}y \cap C)$ = $\overline{B}y \wedge \vee_{x\in f^{-1}y\cap C} L_f \overline{C}x$.

On the other hand, $\overline{E}y = \wedge_{j\in J} \overline{E}_j y$ = $\wedge_{j\in J} (\overline{B}y \wedge \vee L_f \overline{C}_j (f^{-1}y \cap C_j))$.

But by 3.1.1(3), $\wedge_{j\in J} (\overline{B}y \wedge \vee L_f \overline{C}_j (f^{-1}y \cap C_j))$ = $\overline{B}y \wedge \wedge_{j\in J} \vee L_f \overline{C}_j (f^{-1}y \cap C_j)$, implying

$\overline{E}y$ = $\overline{B}y \wedge \wedge_{j\in J} \vee L_f \overline{C}_j (f^{-1}y \cap C_j)$ = $\overline{B}y \wedge \wedge_{j\in J} \vee_{x\in f^{-1}y\cap C_j} L_f \overline{C}_j x$.

Next, for all $x \in f^{-1}y \cap C$, $x \in f^{-1}y \cap C_j$ for all $j \in J$ and $\overline{C}x \le \overline{C}_j x$, implying

$L_f \overline{C}x \le L_f \overline{C}_j x$ $\le$ $\vee_{x\in f^{-1}y\cap C} L_f \overline{C}_j x$ $\le$ $\vee_{x\in f^{-1}y\cap C_j} L_f \overline{C}_j x$ for all $j \in J$ which in turn implies $L_f \overline{C}x \le \wedge_{j\in J} (\vee_{x\in f^{-1}y\cap C_j} L_f \overline{C}_j x)$ for all $x \in f^{-1}y \cap C$ which finally implies $\vee_{x\in f^{-1}y\cap C} L_f \overline{C}x$ $\le \wedge_{j\in J} (\vee_{x\in f^{-1}y\cap C_j} L_f \overline{C}_j x)$.

Therefore $\overline{D}y$ = $\overline{B}y \wedge \vee_{x\in f^{-1}y\cap C} L_f \overline{C}x$ $\le$ $\overline{B}y \wedge \wedge_{j\in J} (\vee_{x\in f^{-1}y\cap C_j} L_f \overline{C}_j x)$ = $\overline{E}y$ for all $y \in D$, implying $\overline{D} \le \overline{E} \mid D$ or $\mathsf{D} \subseteq \mathsf{E}$.

Proposition 5.22: For any 0-p and 0-r f-map $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ and for any family of f-subsets $(\mathsf{C}_j)_{j\in J}$ of B ,we have $\mathsf{F}_*^{-1}(\cup_{j\in J} \mathsf{C}_j)$ = $\cup_{j\in J} \mathsf{F}_*^{-1} \mathsf{C}_j$, whenever

(a) L_B is a finite chain, L_A is a complete infinite meet distributive lattice.

(b) C_j is L_f-regular for each $j \in J$ and * = i or d or p.

Proof: Let C = $\cup_{j\in J} \mathsf{C}_j$. Then C = $\cup_{j\in J} C_j$, L_C = $\vee_{j\in J} L_{C_j}$ = $(\cup_{j\in J} L_{C_j})_{L_B}$ and

$\overline{C}b$ = $\vee_{j\in I_b} \overline{C}_j b$, where I_b = $\{j \in J \mid b \in C_j\}$, for all $b \in C$.

Let D = $\mathsf{F}^{-1}\mathsf{C}$. Then $D = f^{-1}C$, $L_D = L_f^{-1} L_C$ and $\overline{D}a$ = $\overline{A}a \wedge \vee L_f^{-1} \overline{C} fa$, for all $a \in D$.

Let E_j = $\mathsf{F}^{-1}\mathsf{C}_j$. Then E_j = $f^{-1}C_j$, L_{E_j} = $L_f^{-1} L_{C_j}$ and $\overline{E}_j a$ = $\overline{A}a \wedge \vee L_f^{-1} \overline{C}_j fa$, for all $a \in E_j$.

Let E = $\cup_{j\in J} \mathsf{E}_j$. Then E = $\cup_{j\in J} E_j$, $L_E = \vee_{j\in J} L_{E_j}$ = $(\cup_{j\in J} L_{E_j})$ and $\overline{E}a = \vee_{j\in I_a} \overline{E}_j a$, Where I_a = $\{j \in J \mid a \in E_j\}$, for all $a \in E$.

We show that D = E or (1) $\mathsf{D} = \mathsf{E}$ (2) $L_D = L_E$ and (3) $\overline{D} = \overline{E}$.

a. D = $f^{-1}C$ = $f^{-1}(\cup_{j\in J} C_j)$ = $\cup_{j\in J} f^{-1}C_j$ =

$\cup_{j\in J} E_j = E$.

b. By 3.2.3(3), $L_{C_j} = [0,\beta_j]$ for some $\beta_j \in L_B$ and for each $j \in J$.

By 3.2.3(8)(b), $\vee_{j\in J} L_{C_j} = \vee_{j\in J}[0,\beta_j] = [0,\vee_{j\in J}\beta_j]$.

Next, since (i) F and hence L_f is 0-p and (ii) C_j is L_f-regular and hence $\beta_j \in L_{C_j} \subseteq L_f L_A$, by 3.4.6(3), $L_f^{-1}L_{C_j} = L_f^{-1}[0,\beta_j] = [0,\vee L_f^{-1}\beta_j]$.

Since $\vee_{j\in J}\beta_j \in L_f L_A$, again as above $L_D = L_f^{-1}L_C = L_f^{-1}(\vee_{j\in J} L_{C_j}) = L_f^{-1}[0,\vee_{j\in J}\beta_j] = [0,\vee L_f^{-1}(\vee_{j\in J}\beta_j)]$.

But since L_f is 0-r, $\beta_j \in L_f L_A$ and L_B is a finite chain, by 3.3.19, $\vee L_f^{-1}(\vee_{j\in J}\beta_j) = \vee_{j\in J}\vee L_f^{-1}\beta_j$ and we get from the above that $L_D = [0,\vee_{j\in J}\vee L_f^{-1}\beta_j]$.

On the other hand, again 3.4.6(3) and 3.2.3(8)(b) as above imply $L_E = \vee_{j\in J} L_{E_j} = \vee_{j\in J} L_f^{-1}L_{C_j}$

$= \vee_{j\in J} L_f^{-1}[0,\beta_j] = \vee_{j\in J}[0,\vee L_f^{-1}\beta_j] = [0,\vee_{j\in J}\vee L_f^{-1}\beta_j]$, since L_f is 0-p. Clearly, now $L_D = L_E$.

c. Let $x \in D = E$ be fixed. Then $\overline{D}x = \overline{A}x \wedge \vee L_f^{-1}\overline{C}fx = \overline{A}x \wedge \vee L_f^{-1}(\vee_{j\in I_{fx}}\overline{C}_j fx) = \overline{A}x \wedge \vee_{j\in I_{fx}} \vee L_f^{-1}\overline{C}_j fx$,

where the last equality is due to 3.3.19, since (i) L_B is a finite chain and (ii) L_F is 0-r.

On the other hand, since L_A is a complete infinite meet distributive lattice,

$\overline{E}x = \vee_{j\in I_x}\overline{E}_j x = \vee_{j\in I_x}(\overline{A}x \wedge \vee L_f^{-1}\overline{C}_j fx) = \overline{A}x \wedge \vee_{j\in I_x}\vee L_f^{-1}\overline{C}_j fx$, where $I_x = \{j \in J \mid x \in E_j\}$.

From the above, clearly it is enough to show that $\vee_{j\in I_{fx}}\vee L_f^{-1}\overline{C}_j fx = \vee_{k\in I_x}\vee L_f^{-1}\overline{C}_k fx$, where

$I_{fx} = \{j \in J \mid fx \in C_j\}$, $I_x = \{k \in J \mid x \in E_k = f^{-1}C_k\}$.

But in order for the equality it is enough to show that $I_{fx} = I_x$.

Let $j \in I_{fx}$. Then $fx \in C_j$ which implies $x \in f^{-1}C_j = E_j$, implying $j \in I_x$.

Conversely, $k \in I_x$ implies $x \in E_k = f^{-1}C_k$ which implies $fx \in C_k$ implying $k \in I_{fx}$.

Therefore $I_{fx} = I_x$ and hence $\overline{D} = \overline{E}$ or D = E.

The following example shows that the proposition is *not* true if some C_j is not L_f-regular:

Example 5.23: Let $F: A \to B$ be given by: A = $(\{a\}, \{(a,1)\}, \{0,\alpha,1 \mid 0<\alpha<1\})$, B = $(\{b\}, \{(b,1)\}, \{0,\alpha,\beta,1 \mid 0<\alpha<\beta<1\})$, $f = \{(a,b)\}$, $L_f = \{(0,0), (\alpha,\alpha), (1,1)\}$.

Let $C_1 = (\{b\},\{b,\alpha\},\{0,\alpha \mid 0<\alpha\})$ and $C_2 = (\{b\},\{(b,\beta)\},\{0,\alpha,\beta \mid 0<\alpha<\beta\})$.

Then $\overline{B}fa = 1 = L_f\overline{A}a$, implies F is preserving 0-p and 0-r. $L_{C_1} = \{0,\alpha\} \subseteq L_f L_A = \{0,\alpha,1\}$, implies C_1 is L_f-regular, but $L_{C_2} = \{0,\alpha,\beta\} \not\subseteq L_f L_A = \{0,\alpha,1\}$, implies C_2 is *not* L_f-regular.

Further, if C = $\cup_{j=1,2} C_j$, then $C = \{b\}$, $L_C = \vee_{j\in J} L_{C_j} = L_{C_1} \vee L_{C_2} = L_{C_2} = \{0,\alpha,\beta\}$ and $\overline{C}b = \vee_{j\in I_b}\overline{C}_j b = \overline{C}_1 b \vee \overline{C}_2 b = \overline{C}_2 b = \beta$.

Let D = $F_p^{-1}C$. Then $D = f^{-1}C = \{a\}$, $L_D = L_f^{-1}L_C = \{0,\alpha\}$ and $\overline{D}a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}fa = 1 \wedge \vee\phi = 1 \wedge 0 = 0$.

Let $E_j = F_p^{-1}C_j$. Then $E_1 = f^{-1}C_1 = \{a\}$, $E_2 = f^{-1}C_2 = \{a\}$, $L_{E_1} = L_f^{-1}L_{C_1} = \{0,\alpha\}$, $L_{E_2} = L_f^{-1}L_{C_2} = \{0,\alpha\}$, $\overline{E}_1 a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}_1 fa = 1 \wedge \alpha = \alpha$ and $\overline{E}_2 a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}_2 fa = 1 \wedge \vee\phi = 0$.

Let E = $\cup_{j\in J} E_j$. Then $E = E_1 \cup E_2 = \{a\} = D$, $L_E = L_{E_1} \vee L_{E_2} = L_D$ and $\overline{E}a = \overline{E}_1 a \vee \overline{E}_2 a = \alpha \vee 0 = \alpha \neq 0 = \overline{D}a$, implying $D \neq E$.

The following example shows that the Proposition is *not* true if L_B is *not* a finite chain:

Example 5.24: Let $F: A \to B$ be defined by: A = $(\{a\},\{(a,1)\},[0,1])$, B = $(\{b\},\{(b,1)\},[0,1])$, $f = \{(a,b)\}$ and $L_f = \{(x,0) \mid x \in [0,1/3]\} \cup \{(x,3(x-1/3)) \mid x \in [1/3,2/3]\} \cup \{(x,1) \mid x \in [2/3,1]\}$.

Then $\overline{B}fa = 1 = L_f\overline{A}a$, implies F is preserving.

Let $\beta_n = 1-1/n, n \geq 1$ and $B_n = (\{b\},\{(b,\beta_n)\},[0,\beta_n])$.

Let $A_n = F_p^{-1}B_n$. Then $A_n = f^{-1}B_n = \{a\}, L_{A_n} =$

$L_f^{-1}L_{B_n} = L_f^{-1}[0,\beta_n] = [0,\vee L_f^{-1}\beta_n] = [0,\alpha_n]$, $\alpha_n < 2/3$ and $\overline{A}_n a = \overline{A}a \wedge \vee L_f^{-1}\overline{B}_n fa = 1 \wedge \alpha_n = \alpha_n < 2/3$ for all $n \geq 1$.

Let $\mathsf{D} = \cup \mathsf{B}_n$. Then $D = \cup B_n = \{b\}$, $L_D = \vee L_{B_n} = \vee[0,\beta_n] = [0,\vee\beta_n] = [0,1]$ and $\overline{D}b = \vee \overline{B}_n b = \vee \beta_n = 1$.

Let $\mathsf{E} = \cup \mathsf{A}_n$. Then $E = \cup A_n = \{a\}$, $L_E = \vee L_{A_n} = \vee[0,\alpha_n] = [0,\vee\alpha_n] = [0,2/3]$ and $\overline{E}a = \vee \overline{A}_n a = 2/3$.

Let $\mathsf{C} = \mathsf{F}_p^{-1}\mathsf{D}$. Then $C = f^{-1}D = \{a\} = E$, $L_C = L_f^{-1}L_D = L_f^{-1}[0,1] = [0,1] \neq L_E = [0,2/3]$ and $\overline{C}a = \overline{A}a \wedge \vee L_f^{-1}\overline{D}fa = 1 \wedge 1 = 1 \neq 2/3 = \overline{E}a$, implying $\mathsf{C} \neq \mathsf{E}$.

Proposition 5.25: For any 0-p and 1-p f-map $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ and for any family of f-subsets $(\mathsf{C}_j)_{j\in J}$ of B, we have

$\mathsf{F}_*^{-1}(\cap_{j\in J}\mathsf{C}_j) = \cap_{j\in J}\mathsf{F}_*^{-1}\mathsf{C}_j$, whenever C_j is L_f-regular for each j $\in$ J and * = i or d or p.

***Proof* :** Let $\mathsf{C} = \cap_{j\in J}\mathsf{C}_j$. Then $C = \cap_{j\in J} C_j$, $L_C = \wedge_{j\in J} L_{C_j} = \cap_{j\in J} L_{C_j}$ and $\overline{C}c = \wedge_{j\in J}\overline{C}_j c$, for all $c \in C$.

Let $\mathsf{D} = \mathsf{F}^{-1}\mathsf{C}$. Then $D = f^{-1}C$, $L_D = L_f^{-1}L_C$ and $\overline{D}a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}fa$, for all $a \in D$.

Let $\mathsf{E}_j = \mathsf{F}^{-1}\mathsf{C}_j$. Then $E_j = f^{-1}C_j$, $L_{E_j} = L_f^{-1}L_{C_j}$ and $\overline{E}_j a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}_j fa$, for all $a \in E_j$.

Let $\mathsf{E} = \cap_{j\in J}\mathsf{E}_j$. Then $E = \cap_{j\in J} E_j$, $L_E = \wedge_{j\in J} L_{E_j} = \cap_{j\in J} L_{E_j}$ and $\overline{E}a = \wedge_{j\in J}\overline{E}_j a$, for all $a \in E$.

From the above it is enough to show that $\mathsf{D} = \mathsf{E}$ or (1) $D = E$ (2) $L_D = L_E$ and (3) $\overline{D} = \overline{E}$.

a. $D = f^{-1}C = f^{-1}(\cap_{j\in J} C_j) = \cap_{j\in J} f^{-1}C_j = \cap_{j\in J} E_j = E$.

b. By 3.2.3(3), $L_{C_j} = [0,\beta_j]$ for some $\beta_j \in L_B$ and for each $j \in J$.

By 3.2.3(8)(a), $\wedge_{j\in J} L_{C_j} = \wedge_{j\in J}[0,\beta_j] = [0,\wedge_{j\in J}\beta_j]$.

Next, since (i) F and hence L_f is 0-p and (ii) C_j is L_f-regular and hence $\beta_j \in L_{C_j} \subseteq L_f L_A$,

by 3.4.6(3), $L_f^{-1}L_{C_j} = L_f^{-1}[0,\beta_j] = [0,\vee L_f^{-1}\beta_j]$.

Since $\wedge_{j\in J}\beta_j \in L_f L_A$, again as above $L_D = L_f^{-1}L_C = L_f^{-1}(\wedge_{j\in J} L_{C_j}) = L_f^{-1}[0,\wedge_{j\in J}\beta_j] = [0,\vee L_f^{-1}(\wedge_{j\in J}\beta_j)]$, by 3.4.6(3).

But since F and hence L_f is 1-p, $\beta_j \in L_f L_A$ for all $j \in J$, by 3.3.16, we get that

$\vee L_f^{-1}(\wedge_{j\in J}\beta_j) = \wedge_{j\in J} \vee L_f^{-1}\beta_j$, and from the above that $L_D = [0,\wedge_{j\in J} \vee L_f^{-1}\beta_j]$.

On the other hand, again 3.2.3(8)(a) with the above implies $L_E = \wedge_{j\in J} L_{E_j} = \wedge_{j\in J}(L_f^{-1}L_{C_j}) = \wedge_{j\in J}[0,\vee L_f^{-1}\beta_j] = [0,\wedge_{j\in J} \vee L_f^{-1}\beta_j]$, implying that $L_D = L_E$.

c. Let $x \in D = E$ be fixed. Then $\overline{D}x = \overline{A}x \wedge \vee L_f^{-1}\overline{C}fx = \overline{A}x \wedge \vee L_f^{-1}(\wedge_{j\in J}\overline{C}_j fx) = \overline{A}x \wedge \wedge_{j\in J} \vee L_f^{-1}\overline{C}_j fx$, where the last equality is due to 3.3.16, since (i) L_f is 1-p and

(ii) $T = \{\overline{C}_j fx \mid j \in J\} \subseteq \cup_{j\in J} L_{C_j} \subseteq L_f L_A$

because each C_j is L_f-regular.

On the other hand, by 3.1.1(3), $\overline{E}x = \wedge_{j\in J}\overline{E}_j x = \wedge_{j\in J}(\overline{A}x \wedge \vee L_f^{-1}\overline{C}_j fx) = \overline{A}x \wedge \wedge_{j\in J} \vee L_f^{-1}\overline{C}_j fx$, implying $\overline{D}x = \overline{E}x$.

The following example shows that the Proposition is *not* true if some C_j is *not* L_f-regular:

Example 5.26: *Let* $\mathsf{F}: \mathsf{A} \to \mathsf{B}$ *be defined by:* $\mathsf{A} = (\{a\}, \{(a,1)\}, \{0,\alpha,1 \mid 0 < \alpha < 1\})$, $\mathsf{B} = (\{b\}, \{b,1\}, \{0,\alpha,\beta,1 \mid 0 < \alpha < \beta < 1\})$, $f = \{(a,b)\}$ and $L_f = \{(0,0),(\alpha,\alpha),(1,1)\}$.

Let $\mathsf{C}_1 = (\{b\}, \{(b,\alpha)\}, \{0,\alpha \mid 0 < \alpha\})$ and $\mathsf{C}_2 = (\{b\}, \{(b,\beta)\}, \{0,\alpha,\beta \mid 0 < \alpha < \beta\})$.

Then $\overline{B}fa = 1 = L_f\overline{A}a$, implying F is preserving, $L_{C_1} = \{0,\alpha\} \subseteq L_f L_A = \{0,\alpha,1\}$, implying F is 0-p and 1-p, C_1 is L_f-regular, and $L_{C_2} = \{0,\alpha,\beta\} \not\subseteq L_f L_A = \{0,\alpha,1\}$, implying C_2 is *not* L_f-regular.

Let $\mathsf{C} = \mathsf{C}_1 \cap \mathsf{C}_2$. Then $C = C_1 \cap C_2 = \{b\}$, $L_C = L_{C_1} \wedge L_{C_2} = \{0,\alpha\} = L_{C_1}$ and $\overline{C} = \overline{C}_1 \wedge \overline{C}_2 = \{(b,\alpha)\} \wedge \{(b,\beta)\} = \{(b,\alpha\wedge\beta)\} = \{(b,\alpha)\} = \overline{C}_1$.

Let $\mathsf{D} = \mathsf{F}_p^{-1}\mathsf{C}$. Then $D = f^{-1}C = \{a\}$, $L_D = L_f^{-1}L_C = L_f^{-1}\{0,\alpha\} = \{0,\alpha\}$ and $\overline{D}a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}fa = 1 \wedge \alpha = \alpha$.

Let $\mathsf{E}_j = \mathsf{F}^{-1}\mathsf{C}_j$. Then $E_1 = f^{-1}C_1$, $E_2 = f^{-1}C_2 = \{a\}$, $L_{E_1} = L_f^{-1}L_{C_1} = \{0,\alpha\}$, $L_{E_2} = L_f^{-1}L_{C_2} = \{0,\alpha\}$, $\overline{E}_1 a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}_1 fa = 1 \wedge \alpha = \alpha$ and $\overline{E}_2 a = \overline{A}a \wedge \vee L_f^{-1}\overline{C}_2 fa = 1 \wedge \vee\phi = 1 \wedge 0 = 0$.

Let $\mathsf{E} = \cap_{j=1,2}\mathsf{E}_j$. Then $E = E_1 \cap E_2 = \{a\} = D$, $L_E = L_{E_1} \wedge L_{E_2} = L_D$ and $\overline{E}a = \overline{E}_1 a \wedge \overline{E}_2 a = \alpha \wedge 0 = 0 \neq \alpha = \overline{D}a$, implying $\mathsf{D} \neq \mathsf{E}$.

Proposition 5.27: For any pair of f-maps $\mathsf{F}:\mathsf{A}\to\mathsf{B}$ and $\mathsf{G}:\mathsf{B}\to\mathsf{C}$ and for any f-subset E of A, the following are true:

(a) $(\mathsf{G}_*\mathsf{F}_i)(\mathsf{E}) = \mathsf{G}_*(\mathsf{F}_i\mathsf{E})$

(b) $(\mathsf{G}_d\mathsf{F}_*)\mathsf{E} = \mathsf{G}_d(\mathsf{F}_*\mathsf{E})$, when L_C is a complete infinite meet distributive lattice

(c) $(\mathsf{G}_p\mathsf{F}_p)\mathsf{E} = \mathsf{G}_p(\mathsf{F}_p\mathsf{E})$, when L_C is a complete infinite meet distributive lattices.

Proof: Let $(\mathsf{GF})\mathsf{E}$ = H. Then $H = gfE$, $L_H = (L_g L_f L_E)_{L_C}$ and $\overline{H}c = \overline{C}c \wedge \vee L_g L_f \overline{E}((gf)^{-1}c \cap E)$ for all $c \in H$.

Let FE = I. Then $I = fE$, $L_I = (L_f L_E)_{L_B}$ and $\overline{I}b = \overline{B}b \wedge \vee L_f \overline{E}(f^{-1}b \cap E)$ for all $b \in I$.

Let GI = K. Then $K = gI$, $L_K = (L_g L_I)_{L_C}$ and $\overline{K}c = \overline{C}c \wedge \vee L_g \overline{I}(g^{-1}c \cap I)$ for all $c \in K$.

(a): From the above it is enough to show that H = K or (1) $H = K$ (2) $L_H = L_K$ and (3) $\overline{H} = \overline{K}$.

a. $H = gfE = g(fE) = gI = K$.

b. By 3.2.3(3), $L_E = [0,\alpha]$ for some $\alpha \in L_A$. By 3.4.3(2), $L_I = (L_f L_E)_{L_B} = (L_f[0,\alpha])_{L_B} = [0, L_f\alpha]$. Again by 3.4.3(2), $L_K = (L_g L_I)_{L_C} = (L_g[0,L_f\alpha])_{L_C} = [0, L_g L_f \alpha]$.

On the other hand, again by 3.4.3(2), $L_H = (L_g L_f L_E)_{L_C} = (L_g L_f[0,\alpha])_{L_C} = [0, L_g L_f\alpha]$. Clearly, $L_K = L_H$.

c. Let $y \in I$. Since F is increasing and $\mathsf{E} \subseteq \mathsf{A}$, $\overline{B}f \geq L_f\overline{A} \geq L_f\overline{E}$. For any $x \in f^{-1}y \cap E$, $fx = y$ and $L_f\overline{E}x \leq L_f\overline{A}x \leq \overline{B}fx = \overline{B}y$, implying $\vee L_f \overline{E}(f^{-1}y \cap E) \leq \overline{B}y$ or $\overline{I}y = \overline{B}y \wedge \vee L_f\overline{E}(f^{-1}y\cap E) = \vee L_f \overline{E}(f^{-1}y \cap E)$ for all $y \in I$.

Let $z \in H = K$ be fixed. Then $\overline{H}z = \overline{C}z \wedge \vee L_g L_f \overline{E}((gf)^{-1}z \cap E)$ and $\overline{K}z = \overline{C}z \wedge \vee L_g \overline{I}(g^{-1}z\cap I) = \overline{C}z \wedge \vee_{y\in g^{-1}z\cap I} L_g \overline{I}y$.

Since (i) $z \in H$ implies $z = gfx$ for some $x \in E$, implying: (a) $x \in (gf)^{-1}z \cap E$ implying $(gf)^{-1}z \cap E \neq \phi$ (b) $y = fx \in g^{-1}z \cap I$ implying $g^{-1}z \cap I \neq \phi$ and (c) $x \in f^{-1}y \cap E$ implying $f^{-1}y \cap E \neq \phi$ (ii) F is increasing (iii) $\mathsf{E} \subseteq \mathsf{A}$ (iv) $(gf)^{-1}z \cap E = \cup_{y\in g^{-1}z\cap fE} f^{-1}y \cap E$ and

(v) $\vee_{\alpha\in\cup_{i\in I}A_i}\alpha = \vee_{i\in I}\vee_{\alpha\in A_i}\alpha$, we get that $\overline{K}z = \overline{C}z \wedge \vee_{y\in g^{-1}z\cap I} L_g(\vee_{x\in f^{-1}y\cap E} L_f\overline{E}x) = \overline{C}z \wedge \vee_{y\in g^{-1}z\cap I}\vee_{x\in f^{-1}y\cap E} L_g L_f \overline{E}x = \overline{C}z \wedge \vee_{x\in\cup_{y\in g^{-1}z\cap fE} f^{-1}y\cap E} L_g L_f\overline{E}x = \overline{C}z \wedge \vee_{x\in((gf)^{-1}z\cap E)} L_g L_f \overline{E}x = \overline{C}z \wedge \vee L_g L_f\overline{E}((gf)^{-1}z\cap E) = \overline{H}z$.

(b): Let H,I and K be as in (a) above. Then it is enough to show, when G is decreasing, that H = K or

a. $H = K$ (2) $L_H = L_K$ and (3) $\overline{H} = \overline{K}$.

b. $H = K$ as in (a) above.

c. $L_H = L_K$ again as in (a) above.

d. Let $z \in H = K$ be fixed. Then $\overline{H}z = \overline{C}z \wedge \vee L_g L_f \overline{E}((gf)^{-1}z \cap E)$ and $\overline{K}z = \overline{C}z \wedge \vee L_g \overline{I}(g^{-1}z \cap I) = \overline{C}z \wedge \vee_{y\in g^{-1}z\cap I} L_g\overline{I}y$.

Since G is decreasing, $\overline{C}g \leq L_g\overline{B}$. So, for each $y \in g^{-1}z \cap I$, $gy = z$, $y \in I$ and $\overline{C}z = \overline{C}gy \leq L_g\overline{B}y$, implying $\overline{C}z \wedge L_g\overline{B}y = \overline{C}z$.

Let $c = \overline{C}z$, $a_y = L_g\overline{B}y$, $b_y = \vee_{x\in f^{-1}y\cap E} L_g L_f \overline{E}x$ and $Y = g^{-1}z \cap I$.

Again since (i) $z \in H$ implies $z = gfx$ for some $x \in E$, implying: (a) $x \in (gf)^{-1}z \cap E$ implying $(gf)^{-1}z \cap E \neq \phi$ (b) $y = fx \in g^{-1}z \cap I$ implying $g^{-1}z \cap I \neq \phi$ and (c) $x \in f^{-1}y \cap E$ implying $f^{-1}y \cap E \neq \phi$ (ii) L_C is a complete infinite meet distributive lattice (iii) $(gf)^{-1}z \cap E = \cup_{y\in g^{-1}z\cap fE} f^{-1}y \cap E$ and (iv) $\vee_{\alpha\in\cup_{i\in I}A_i}\alpha = \vee_{i\in I}\vee_{\alpha\in A_i}\alpha$, from the above we get that

$\overline{K}z = \overline{C}z \wedge \vee_{y\in g^{-1}z\cap I} L_g(\overline{B}y \wedge \vee_{x\in f^{-1}y\cap E} L_f \overline{E}x) = \overline{C}z \wedge \vee_{y\in g^{-1}z\cap I}(L_g \overline{B}y \wedge L_g(\vee_{x\in f^{-1}y\cap E} L_f \overline{E}x))$

$= \overline{C}z \wedge \vee_{y\in g^{-1}z\cap I}(L_g \overline{B}y \wedge \vee_{x\in f^{-1}y\cap E} L_g L_f \overline{E}x) = c \wedge \vee_{y\in Y}(a_y \wedge b_y) = \vee_{y\in Y}(c \wedge a_y \wedge b_y)$

$= \vee_{y\in Y}(c \wedge b_y) = c \wedge \vee_{y\in Y} b_y = \overline{C}z \wedge \vee_{y\in g^{-1}z\cap I} \vee_{x\in f^{-1}y\cap E} L_g L_f \overline{E}x$

$= \overline{C}z \wedge \vee_{x\in \cup_{y\in g^{-1}z\cap I} f^{-1}y\cap E} L_g L_f \overline{E}x = \overline{C}z \wedge \vee_{x\in (gf)^{-1}z\cap E} L_g L_f \overline{E}x$

$= \overline{C}z \wedge \vee L_g L_f \overline{E}((gf)^{-1}z \cap E) = \overline{H}z$, implying $\overline{K}z = \overline{H}z$.

(c): Clearly, the proof follows from (a) and (b).

Proposition 5.28 : *For any pair of f-maps* $\mathsf{F}:\mathsf{A}\to\mathsf{B}$ *and* $\mathsf{G}:\mathsf{B}\to\mathsf{C}$ *and for any f-subset* E *of* C*, the following are true:*

(a) $(\mathsf{G}_d\mathsf{F}_*)^{-1}\mathsf{E} \supseteq \mathsf{F}_*^{-1}(\mathsf{G}_d^{-1}\mathsf{E})$, whenever E is L_g-regular

(b) $(\mathsf{G}_*\mathsf{F}_i)^{-1}\mathsf{E} \subseteq \mathsf{F}_i^{-1}(\mathsf{G}_*^{-1}\mathsf{E})$, whenever $\mathsf{G}^{-1}\mathsf{E}$ is L_f-regular and F is 0-p

(c) $(\mathsf{G}_p\mathsf{F}_p)^{-1}\mathsf{E} = \mathsf{F}_p^{-1}(\mathsf{G}_p^{-1}\mathsf{E})$, whenever $\mathsf{G}^{-1}\mathsf{E}$ is L_f-regular and E is L_g-regular and F is 0-p.

Proof: Let $(\mathsf{GF})^{-1}\mathsf{E} = \mathsf{H}$. Then $H = (gf)^{-1}E$, $L_H = (L_gL_f)^{-1}L_E$ and $\overline{H}a = \overline{A}a \wedge \vee(L_gL_f)^{-1}\overline{E}(gf)a$ for all $a\in H$.

Let $\mathsf{G}^{-1}\mathsf{E} = \mathsf{I}$. Then $I = g^{-1}E$, $L_I = L_g^{-1}L_E$ and $\overline{I}b = \overline{B}b \wedge \vee L_g^{-1}\overline{E}gb$ for all $b\in I$.

Let $\mathsf{F}^{-1}\mathsf{I} = \mathsf{K}$. Then $K = f^{-1}I$, $L_K = L_f^{-1}L_I$ and $\overline{K}a = \overline{A}a \wedge \vee L_f^{-1}\overline{I}fa$ for all $a\in K$.

From the above it is enough to show that $\mathsf{H}\supseteq\mathsf{K}$ or (a) $K\subseteq H$ (b) L_K is a complete ideal of L_H and (c) $\overline{K} \le \overline{H}|K$.

a. $K = f^{-1}I = f^{-1}g^{-1}E = H$.

b. $L_K = L_f^{-1}L_I = L_f^{-1}L_g^{-1}L_E = L_H$.

c. Let $a\in f^{-1}g^{-1}E = H = K$ be fixed. Then $gfa\in E$, $fa\in g^{-1}E = I$, $\overline{H}a = \overline{A}a \wedge \vee L_f^{-1}L_g^{-1}\overline{E}gfa$ and $\overline{K}a = \overline{A}a \wedge \vee L_f^{-1}\overline{I}fa = \overline{A}a \wedge \vee L_f^{-1}(\overline{B}fa \wedge \vee L_g^{-1}\overline{E}gfa)$.

Firstly, E is L_g-regular implies $L_E \subseteq L_gL_B$; $\overline{E}gfa \in L_E \subseteq L_gL_B$ implies $\overline{E}gfa \in L_gL_B$; so, by 3.3.11(3), $L_g(\vee L_g^{-1}\overline{E}gfa) = \overline{E}gfa$.

Since G is decreasing and $\mathsf{E}\subseteq\mathsf{C}$ we have $\overline{E}gfa \le \overline{C}gfa \le L_g\overline{B}fa$

$L_g\overline{I}fa = L_g\overline{B}fa \wedge L_g(\vee L_g^{-1}\overline{E}gfa) = L_g\overline{B}fa \wedge \overline{E}gfa = \overline{E}gfa$, implying $\overline{I}fa \in L_g^{-1}\overline{E}gfa$ which implies $L_f^{-1}\overline{I}fa \subseteq L_f^{-1}L_g^{-1}\overline{E}gfa$ which in turn implies $\vee L_f^{-1}\overline{I}fa \le \vee L_f^{-1}L_g^{-1}\overline{E}gfa$ or $\overline{K}a = \overline{A}a \wedge \vee L_f^{-1}\overline{I}fa \le \overline{A}a \wedge \vee L_f^{-1}L_g^{-1}\overline{E}gfa = \overline{H}a$.

(b): Let H,I and K be as in (a) above. Then it is enough to show, when F is increasing and 0-p and when $\mathsf{G}^{-1}\mathsf{E}$ is L_f-regular, that $\mathsf{H} \subseteq \mathsf{K}$ or (1) $H\subseteq K$ (2) L_H is a complete ideal of L_K and (3) $\overline{H} \le \overline{K}|H$.

(a): $H = K$ as in (a) above.

(b): $L_H = L_K$ again as in (a) above.

(c): Let $a\in H = K = f^{-1}g^{-1}E$ be fixed. Then $gfa\in E$, $fa\in g^{-1}E = I$, $\overline{H}a = \overline{A}a \wedge \vee L_f^{-1}L_g^{-1}\overline{E}gfa$ and $\overline{K}a = \overline{A}a \wedge \vee L_f^{-1}\overline{I}fa = \overline{A}a \wedge \vee L_f^{-1}(\overline{B}fa \wedge \vee L_g^{-1}\overline{E}gfa)$.

$gfa\in E$ implies $\overline{E}gfa \in \overline{E}E \subseteq L_E$ which implies $L_g^{-1}\overline{E}gfa \subseteq L_g^{-1}L_E = L_I \subseteq L_fL_A$, since $\mathsf{G}^{-1}\mathsf{E} = \mathsf{I}$ is L_f-regular.

Since L_f is 0-p and $D = L_g^{-1}\overline{E}gfa \subseteq L_fL_A$, by 3.3.9, $L_f(\vee L_f^{-1}L_g^{-1}\overline{E}gfa) = \vee L_g^{-1}\overline{E}gfa$ and $L_f\overline{H}a = L_f\overline{A}a \wedge L_F(\vee L_f^{-1}L_g^{-1}\overline{E}gfa) = L_f\overline{A}a \wedge \vee L_g^{-1}\overline{E}gfa \le \overline{B}fa \wedge \vee L_g^{-1}\overline{E}gfa = \overline{I}fa$, where the last inequality is due to the fact that F is increasing and hence $L_f\overline{A} \le \overline{B}f$.

Again $gfa\in E$ implies $fa\in g^{-1}E = I$ which implies $\overline{I}fa \in \overline{I}I \subseteq L_I \subseteq L_fL_A$, since $\mathsf{G}^{-1}\mathsf{E} = \mathsf{I}$ is L_f-regular.

Since $\overline{I}fa \in L_fL_A$ and $L_f\overline{H}a \le \overline{I}fa$, as above by 3.3.2, we get that $\vee L_f^{-1}L_f\overline{H}a \le \vee L_f^{-1}\overline{I}fa$. But then $\overline{H}a \in L_f^{-1}L_f\overline{H}a$ implies $\overline{H}a \le \vee L_f^{-1}L_f\overline{H}a \le \vee L_f^{-1}\overline{I}fa$. Since always $\overline{H}a \le \overline{A}a$, it follows that $\overline{H}a \le \overline{K}a$

(c): Clearly, the proof follows from (a) and (b).

The following example shows that a strict containment inthe conclusion (a) may hold in the above proposition:

Example 5.29: Let $F: A \to B$ and $G: B \to C$ be defined by $A = (\{a\}, \{(a,1)\}, \{0, \alpha_1, \alpha_2, \beta_1, \beta_2, 1 \mid 0 < \alpha_i, \beta_j < 1; \alpha_1 < \alpha_2, \beta_1 < \beta_2; \alpha_i \parallel \beta_j\})$, B $= (\{b\}, \{(b,0)\}, \{0, \alpha, \beta, 1 \mid 0 < \alpha, \beta < 1; \alpha \parallel \beta\})$, $C = (\{c\}, \{(c,0)\}, \{0,1 \mid 0 < 1\})$, $f = \{(a,b)\}$, $g = \{(b,c)\}$, $L_f = \{(0,0),(1,1),(\alpha_i,\alpha),(\beta_i,\beta) \mid i = 1,2\}$, $L_g = \{(\alpha,0),(0,0),(\beta,1),(1,1)\}$ and E $= (\{c\}, \{c,0\}, \{0,1 \mid 0 < 1\})$.

Then $L_E = \{0,1\} = L_g L_B$ implying E is L_g-regular; $\overline{B}fa = 0 \le L_f \overline{A}a = 1$, implying F is decreasing and $\overline{C}gb = 0 = L_g \overline{B}b$, implying G is preserving.

Let $(G_p F_d)^{-1} E = H$. Then $H = (gf)^{-1} E = f^{-1} g^{-1} E = \{a\}$, $L_H = (L_g L_f)^{-1} L_E = L_f^{-1} L_g^{-1} L_E = L_f^{-1}\{0,\alpha,\beta,1\} = L_A$ and $\overline{H}a = \overline{A}a \wedge \vee (L_g L_f)^{-1} \overline{E} gfa = 1 \wedge \alpha_2 = \alpha_2$.

Let $G_p^{-1} E =$ I. Then $I = g^{-1} E = \{b\}$, $L_I = L_g^{-1} L_E = L_B$ and $\overline{I}b = \overline{B}b \wedge \vee L_g^{-1} \overline{E} gb = 0 \wedge \alpha = 0$.

Let $F_d^{-1} I = K$. Then $K = f^{-1} I = \{a\} = H$, $L_K = L_f^{-1} L_I = L_f^{-1} L_B = L_A = L_H$ and $\overline{K}a = \overline{A}a \wedge \vee L_f^{-1} \overline{I} fa = 1 \wedge 0 = 0 < \alpha_2 = \overline{H}a$, implying $\overline{H} > \overline{K}$ or $H \supset K$.

The following example shows that the condition on $G^{-1}E$ is L_f-regular is *not* superfluous in (b) of the above proposition:

Example 5.30: Let $F: A \to B$ and $G: B \to C$ be defined by: $A = (\{a\}, \{(a,\alpha)\}, \{0, \alpha, 1 \mid 0 < \alpha < 1\})$, B $= (\{b\}, \{(b,\beta)\}, \{0, \alpha, \beta, 1 \mid 0 < \alpha < \beta < 1\})$, $C = (\{c\}, \{(c,\delta)\}, \{0, \delta, 1 \mid 0 < \delta < 1\}) =$ E, $f = \{(a,b)\}$, $g = \{(b,c)\}$, $L_f = \{(0,0),(\alpha,\alpha),(1,1)\}$ and $L_g = \{(0,0),(\alpha,\delta),(\beta,\delta),(1,1)\}$.

Then L_f is 0-p, $\overline{B}fa = \beta \ge \alpha = L_f \overline{A}a$ implies F is increasing, $\overline{C}gb = \delta = L_g \overline{B}b$ implies G is preserving, $L_E = \{0,\delta,1\} = L_g L_B$, implying E is L_g-regular and $L_g^{-1} L_E = \{0,\alpha,\beta,1\} \not\subseteq \{0,\alpha,1\} = L_f L_A$, implying $G^{-1}E$ is *not* L_f-regular.

Let $(G_p F_i)^{-1} E =$ H. Then $H = (gf)^{-1} E = f^{-1} g^{-1} E = \{a\}$, $L_H = L_f^{-1} L_g^{-1} L_E = L_f^{-1}\{0,\alpha,\beta,1\} = \{0,\alpha,1\} = L_A$ and $\overline{H}a = \overline{A}a \wedge \vee L_f^{-1} L_g^{-1} \overline{E} gfa = \alpha \wedge \alpha = \alpha$.

Let $G_p^{-1} E =$ I. Then $I = g^{-1} E = \{b\}$, $L_I = L_g^{-1} L_E = \{0,\alpha,\beta,1\} = L_B$ and $\overline{I}b = \overline{B}b \wedge \vee L_g^{-1} \overline{E} gb = \beta \wedge \beta = \beta$.

Let $F_i^{-1} I = K$. Then $K = f^{-1} I = \{a\} = H$, $L_K = L_f^{-1} L_I = L_f^{-1}\{0,\alpha,\beta,1\} = \{0,\alpha,1\} = L_A = L_H$ and $\overline{K}a = \overline{A}a \wedge \vee L_f^{-1} \overline{I} fa = \alpha \wedge \vee \phi = \alpha \wedge 0 = 0 < \alpha = \overline{H}a$, implying $\overline{H} \not\le \overline{K}$ or $H \not\subseteq K$ or $(G_p F_i)^{-1} E \not\subseteq F_i^{-1} G_p^{-1} E$.

The following example shows that the condition on E that it is L_g-regular, is *not* superfluous in (c) of the above proposition:

Example 5.31: Let $F: A \to B$ and $G: B \to C$ be defined by: $A = (\{a\}, \{(a,1)\}, \{0, \alpha, \beta, 1 \mid 0 < \alpha, \beta < 1\})$, B$= (\{b\}, \{(b,1)\}, \{0,1 \mid 0 < 1\})$, $C = (\{c\}, \{(c,1)\}, \{0, \delta, 1 \mid 0 < \delta < 1\})$, $f = \{(a,b)\}$, $g = \{(b,c)\}$, $L_f = \{(0,0),(\alpha,0),(\beta,1),(1,1)\}$, $L_g = \{(0,0),(1,1)\}$ and E $= (\{c\}, \{(c,\delta)\}, \{0, \delta \mid 0 < \delta\})$.

Then $\overline{B}fa = 1 = L_f \overline{A}a$, implies F is preserving; $\overline{C}gb = 1 = L_g \overline{B}b$, implies G is preserving; $L_E = \{0,\delta\} \not\subseteq L_g L_B = \{0,1\}$, implies E is *not* L_g-regular and $L_g^{-1} L_E = \{0\} \subseteq L_f L_A = \{0,1\}$, implies $G_p^{-1} E$ is L_f-regular.

Let $(G_p F_p)^{-1} E =$ H. Then $H = (gf)^{-1} E = f^{-1} g^{-1} E = \{a\}$, $L_H = L_f^{-1} L_g^{-1} L_E = L_f^{-1}(0) = \{0,\alpha\}$ and $\overline{H}a = \overline{A}a \wedge \vee L_f^{-1} L_g^{-1} \overline{E} gfa = 1 \wedge \vee \phi = 1 \wedge 0 = 0$

Let $G_p^{-1} E =$ I. Then $I = g^{-1} E = \{b\}$, $L_I = L_g^{-1} L_E = \{0\}$ and $\overline{I}b = \overline{B}b \wedge \vee L_g^{-1} \overline{E} gb = 1 \wedge \vee \phi = 1 \wedge 0 = 0$.

Let $F_p^{-1} I = K$. Then $K = f^{-1} I = \{a\} = H$, $L_K = L_f^{-1} L_I = L_f^{-1}(0) = \{0,\alpha\} = L_H$ and $\overline{K}a = \overline{A}a \wedge \vee L_f^{-1} \overline{I} fa = 1 \wedge \alpha = \alpha \ne 0 = \overline{H}a$, implying $(G_p F_p)^{-1} E =$ H $\ne K = F_p^{-1} G_p^{-1} E$.

F. *More on f-Images and f-Inverse Images:*

In this section some more standard properties of the M-f-images of L-f-subsets under an f-map and the L-f-inverse images of M-f-subsets under an f-map are studied in detail.

Lemma6.1: *For any 0-p f-map* $\mathsf{F}:\mathsf{A}\to\mathsf{B}$ *and for any* L_f *-regular f-subset* H *of* B*, always* $\mathsf{F}^{-1}\mathsf{H}\supseteq\mathsf{F}^{-1}(\mathsf{H}\cap\mathsf{FA})$ *holds. However, equality holds whenever*

(a) F is increasing, L_f is 1-p and L_B is complete infinite meet distributive lattice (OR)

(b) F is decreasing and L_B is complete infinite meet distributive lattice.

Proof: (A) Since H is L_f -regular and $\mathsf{H}\cap\mathsf{FA}\subseteq\mathsf{H}$, by 4.5.6, F^{-1} is monotonic and so, $\mathsf{F}^{-1}(\mathsf{H}\cap\mathsf{FA})$ $\subseteq$ $\mathsf{F}^{-1}(\mathsf{H})$.

(B) Let $\mathsf{F}^{-1}\mathsf{H}$ = C. Then C = $f^{-1}H$, L_C = $L_f^{-1}L_H$ and $\overline{C}a$ = $\overline{A}a\wedge\vee L_f^{-1}\overline{H}\,fa$ for all $a\in C$.

Let FA = D . Then $D= fA$, L_D = $(L_f L_A)_{L_B}$ and for all $b\in D$, $\overline{D}b$ = $\overline{B}b$ $\wedge$ $\vee L_f\overline{A}(f^{-1}b\cap A)$.

Let $\mathsf{H}\cap\mathsf{D}$ = E. Then E = $H\cap D$, L_E = $L_H\cap L_D$ and $\overline{E}b$ = $\overline{H}b\wedge\overline{D}b$ for all $b\in E$.

Let $\mathsf{F}^{-1}\mathsf{E}=\mathsf{G}$. Then G = $f^{-1}E$, L_G = $L_f^{-1}L_E$ and $\overline{G}a$ = $\overline{A}a\wedge\vee L_f^{-1}\overline{E}\,fa$ for all $a\in G$.

We show that C = G or (1) $C=G$ (2) $L_C=L_G$ (3) $\overline{C}=\overline{G}$ when

(a) F is increasing, L_f is 1-p and L_B is complete infinite meet distributive lattice

(OR)

(b) F is decreasing and L_B is complete infinite meet distributive lattice.

(a): C = $f^{-1}H$ = $f^{-1}(H\cap fA)$ = $f^{-1}(H\cap D)$ = $f^{-1}E$ = G.

(b): By 3.2.3(3), L_H = $[0,\beta]$ for some $\beta\in L_B$. By 3.4.6(3), since (i) F and hence L_f is 0-p

(ii) H is L_f -regular and hence $\beta\in L_H\subseteq L_f L_A$, we get that L_C = $L_f^{-1}L_H$ = $L_f^{-1}[0,\beta]$ = $[0,\vee L_f^{-1}\beta]$.

Since $(L_f L_A)_{L_B}$ is a complete ideal in L_B, the above implies $[0,\beta]\subseteq(L_f L_A)_{L_B}$ which implies

$L_H\cap L_D$ = $[0,\beta]\cap(L_f L_A)_{L_B}$ = $[0,\beta]$ = L_H and L_G = $L_f^{-1}L_E$ = $L_f^{-1}(L_H\cap L_D)$ = $L_f^{-1}L_H$ = L_C.

(c): Let $a\in G$ = $f^{-1}E$ = $C=f^{-1}H$ be fixed. Then $fa\in H\cap E$.

(a): Let F be decreasing. Then $\overline{B}f\le L_f\overline{A}$. Further, for all $c\in f^{-1}fa\cap A$, $L_f\overline{A}c$ $\ge$ $\overline{B}fc$ = $\overline{B}fa$ or $\vee L_f\overline{A}(f^{-1}fa\cap A)$ $\ge$ $\wedge L_f\overline{A}(f^{-1}fa\cap A)$ $\ge$ $\overline{B}fa$, implying $\overline{D}fa$ = $\overline{B}fa\wedge\vee L_f\overline{A}(f^{-1}fa\cap A)$ = $\overline{B}fa$ which in turn implies $\overline{G}a$ = $\overline{A}a\wedge\vee L_f^{-1}\overline{E}\,fa$ = $\overline{A}a\wedge\vee L_f^{-1}(\overline{H}\,fa\wedge\overline{D}\,fa)$ = $\overline{A}a\wedge\vee L_f^{-1}(\overline{H}\,fa\wedge\overline{B}\,fa)$ = $\overline{A}a\wedge\vee L_f^{-1}\overline{H}\,fa$ = $\overline{C}a$, because $\overline{E}$ = $\overline{H}\cap\overline{D}$ and $\overline{H}\le\overline{B}$.

(b): Let F be increasing. Then $\overline{B}f\ge L_f\overline{A}$. For all $c\in f^{-1}fa\cap A$, $L_f\overline{A}c\le\overline{B}fc$ = $\overline{B}fa$ or $\vee L_f\overline{A}(f^{-1}fa\cap A)$ $\le$ $\overline{B}fa$ implying $\overline{D}fa$ = $\overline{B}fa\wedge\vee L_f\overline{A}(f^{-1}fa\cap A)$ = $\vee L_f\overline{A}(f^{-1}fa\cap A)$. Therefore $\overline{E}fa$ = $\overline{H}fa\wedge\overline{D}fa$ = $\overline{H}\,fa\wedge\vee L_f\overline{A}(f^{-1}fa\cap A)$.

Next, since (i) H is L_f -regular and hence $\overline{H}\,fa\in L_H\subseteq L_f L_A$

(ii) $\vee L_f\overline{A}(f^{-1}fa\cap A)$ $\in L_f L_A$ as $f^{-1}fa\cap A\neq\phi$ and

(iii) L_f is 1-p, by 3.3.15, $\vee L_f^{-1}(\overline{H}fa\ \wedge\vee L_f\overline{A}(f^{-1}fa\cap A))=\vee L_f^{-1}\overline{H}\,fa$ $\wedge$ $\vee L_f^{-1}(\vee L_f\overline{A}(f^{-1}fa\cap A))$.

Further, since $\vee L_f\overline{A}(f^{-1}fa\cap A)\in L_f L_A$ as $f^{-1}fa\cap A\neq\phi$ and $\vee L_f\overline{A}(f^{-1}fa\cap A)$ $\ge$ $L_f\overline{A}a$, by 3.3.2, $\vee L_f^{-1}(\vee L_f\overline{A}(f^{-1}fa\cap A))$ $\ge$ $\vee L_f^{-1}(L_f\overline{A}a)$ $\ge$ $\overline{A}a$, where the last inequality is due to the fact that $\overline{A}a\in L_f^{-1}(L_f\overline{A}a)$.

Consequent from the above,

$\overline{G}a$ = $\overline{A}a\wedge\vee L_f^{-1}\overline{E}\,fa$ = $\overline{A}a\wedge\vee L_f^{-1}(\overline{H}\,fa\wedge\vee L_f\overline{A}(f^{-1}fa\cap A))$

= $\overline{A}a\wedge(\vee L_f^{-1}\overline{H}\,fa\wedge\vee L_f^{-1}(\vee L_f\overline{A}(f^{-1}fa\cap A)))$

= $(\overline{A}a\wedge\vee L_f^{-1}(\vee L_f\overline{A}(f^{-1}fa\cap A)))\wedge\vee L_f^{-1}\overline{H}\,fa$ = $\overline{A}a\wedge\vee L_f^{-1}\overline{H}\,fa$ = $\overline{C}a$.

The followingexample showsthat theabove Proposition is *not* true if F is decreasing, L_B is complete infinite meet distributive lattice but H is *not* L_f -regular:

Example 6.2: *Let* $\mathsf{F}:\mathsf{A}\to\mathsf{B}$ *be defined by:* A = $(\{a\},\{(a,1)\},\{a,\alpha,\beta,1\mid 0<\alpha<\beta<1\})$,
B =

$(\{b\},\{(b,\beta)\},\{0,\alpha,\beta,\gamma,1 \mid 0<\alpha<\beta,\gamma<1;\beta \parallel \gamma\})$, $f = \{(a,b)\}$ and $L_f = \{(0,0),(\alpha,\alpha),(\beta,\beta),(1,1)\}$. Let H = $(\{b\},\{(b,\gamma)\},\{0,\alpha,\gamma \mid 0<\alpha<\gamma\})$.

Then L_f is one-one, 0-p and 1-p. $L_H = \{0,\alpha,\gamma\} \not\subseteq L_f L_A = \{0,\alpha,\beta,1\}$ implies H is *not* L_f-regular and $\overline{B}fa = \beta < 1 = L_f\overline{A}a$ implies F is decreasing.

Let $F_d^{-1}H = C$. Then $C = \{a\}$, $L_C = \{0,\alpha\}$ and $\overline{C}a = \overline{A}a \wedge \vee L_f^{-1}\overline{H}fa = 1 \wedge \vee\phi = 1 \wedge 0 = 0$.

Let $F_d A = D$. Then $D = \{b\}$, $L_D = (L_f L_A)_{L_B} = L_B$ and $\overline{D}b = \overline{B}b \wedge \vee L_f \overline{A}(f^{-1}b \cap A) = \beta \wedge 1 = \beta$.

Let $H \cap D = E$. Then $E = H \cap D = \{b\}$, $L_E = L_H \cap L_D = L_H \cap L_B = L_H$ and $\overline{E}b = \overline{H}b \wedge \overline{D}b = \gamma \wedge \beta = \alpha$.

Let $F_d^{-1}E = G$. Then $G = f^{-1}E = \{a\}$, $L_G = L_f^{-1}L_E = \{0,\alpha\}$ and $\overline{G}a = \overline{A}a \wedge \vee L_f^{-1}\overline{E}fa = 1 \wedge \alpha = \alpha \neq 0 = \overline{C}a$, implying $G \neq C$ or $F^{-1}(H \cap B) \neq F^{-1}(H)$.

The following example shows that the above Proposition is *not* true if F is increasing, L_f is 1-p and L_B is complete infinite meet distributive lattice but H is *not* L_f-regular:

Example 6.3: Let $F: A \to B$ be defined by: A = $(\{a\}, \{(a,\beta)\}, \{0,\alpha,\beta,1 \mid 0<\alpha<\beta<1\})$, B= $(\{b\}, \{(b,1)\}, \{0,\alpha,\beta,\gamma,1 \mid 0<\alpha<\beta,\gamma<1, \beta \parallel \gamma\})$, $f = \{(a,b)\}$ and $L_f = \{(0,0), (\alpha,\alpha), (\beta,\beta), (1,1)\}$. Let H = $(\{b\}, \{b,\gamma\}, \{0,\alpha,\gamma \mid 0<\alpha<\gamma\})$.

Then L_B is complete infinite meet distributive lattice, L_f is 1-p $L_H = \{0,\alpha,\gamma\} \not\subseteq L_f L_A = \{0,\alpha,\beta,1\}$, implies H is *not* L_f-regular and $\overline{B}fa = 1 \geq L_f\overline{A}a = \beta$, implies F is increasing.

Let $F_i^{-1}H = C$. Then $C = f^{-1}H = \{a\}$, $L_C = L_f^{-1}L_H = \{(0,\alpha)\}$ and $\overline{C}a = \overline{A}a \wedge \vee L_f^{-1}\overline{H}fa = \beta \wedge \vee\phi = \beta \wedge 0 = 0$.

Let $F_i A = D$. Then $D = fA = \{b\}$, $L_D = (L_f L_A)_{L_B} = L_B$ and $\overline{D}b = \overline{B}b \wedge \vee L_f \overline{A}(f^{-1}b \cap A) = 1 \wedge \beta = \beta$.

Let $H \cap D = E$. Then $E = H \cap D = \{b\}$, $L_E = L_H \cap L_D = L_H \cap L_B = L_H$ and $\overline{E}b = \overline{H}b \wedge \overline{D}b = \gamma \wedge \beta = \alpha$.

Let $F_i^{-1}E = G$. Then $G = f^{-1}E = \{a\}$, $L_G = L_f^{-1}L_E = \{0,\alpha\}$ and $\overline{G}a = \overline{A}a \wedge \vee L_f^{-1}\overline{E}fa = \beta \wedge \alpha = \alpha \neq 0 = \overline{C}a$, implying $G \not\subseteq C$ or $F^{-1}(H \cap B) \neq F^{-1}(H)$.

Lemma 6.4:For any 0-p F:A→B and for any L_f-regular f-subset Y of B, we have $F_*^{-1}F_*F_*^{-1}Y = F_*^{-1}Y$ holds whenever $* = $ i or d or p.

Proof: Let $F^{-1}Y = C$. Then $C = f^{-1}Y$, $L_C = L_f^{-1}L_Y$ and $\overline{C}a = \overline{A}a \wedge \vee L_f^{-1}\overline{Y}fa$ for all $a \in C$.

Let $FC = D$. Then $D = fC$, $L_D = (L_f L_C)_{L_B}$ and for all $b \in D$, $\overline{D}b = \overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C)$.

Let $F^{-1}D = E$. Then $E = f^{-1}D$, $L_E = L_f^{-1}L_D$ and $\overline{E}a = \overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ for all $a \in E$.

We show that E = C or (1) $E = C$ (2) $L_E = L_C$ and (3) $\overline{E} = \overline{C}$.

(a): $E = f^{-1}D = f^{-1}fC = f^{-1}ff^{-1}B = f^{-1}B = C$, since $f^{-1}ff^{-1}B = f^{-1}B$.

(b): By 3.2.3(3), $L_Y = [0,\beta]$ for some $\beta \in L_B$. Since (i) F and hence L_f is 0-p and (ii) Y is L_f-regular and hence $\beta \in L_Y \subseteq L_f L_A$, by 3.4.6(3), $L_C = L_f^{-1}L_Y = L_f^{-1}[0,\beta] = [0, \vee L_f^{-1}\beta]$.

From 3.4.3(2), $L_D = (L_f L_C)_{L_B} = (L_f[0,\vee L_f^{-1}\beta])_{L_B} = [0, L_f(\vee L_f^{-1}\beta)] = [0,\beta] = L_Y$, where the last but one equality follows from 3.3.11(3), since Y is L_f-regular and hence $\beta \in L_Y \subseteq L_f L_A$.

So from the above, $L_E = L_f^{-1}L_D = L_f^{-1}L_Y = L_C$.

(c): Let $a \in E = f^{-1}D = C = f^{-1}Y$ be fixed. Then $fa \in Y \cap D$.

(a): Let F be increasing. Then $\overline{B}f \geq L_f\overline{A}$.

Since $C \subseteq F_*^{-1}F_*C = E$ for all $C \subseteq A$ when $* = i$ or p, we have $\overline{C} \leq \overline{E}$. Therefore it is enough to show that $\overline{E} \leq \overline{C}$.

But since $\overline{E}a = \overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ and $\overline{C}a = \overline{A}a \wedge \vee L_f^{-1}\overline{Y}fa$, it is enough to show that $\vee L_f^{-1}\overline{D}fa \leq \vee L_f^{-1}\overline{Y}fa$.

Let $c \in f^{-1}fa \cap C$. Then $c \in C$ and $fc = fa$.

Further, since Y is L_f-regular, $\overline{Y}fc$ = $\overline{Y}fa \in L_Y \subseteq L_f L_A$ and hence by 3.3.11(3), $L_f(\vee L_f^{-1}\overline{Y}fc)$ = $\overline{Y}fc$ = $\overline{Y}fa$.

Now $L_f\overline{C}c$ = $L_f(\overline{A}c \wedge \vee L_f^{-1}\overline{Y}fc)$ = $L_f\overline{A}c$ $\wedge$ $L_f(\vee L_f^{-1}\overline{Y}fc)$ = $L_f\overline{A}c \wedge \overline{Y}fc$ $\leq$ $\overline{Y}fc$ = $\overline{Y}fa$, implying $\vee L_f\overline{C}(f^{-1}fa \cap c)$ $\leq$ $\overline{Y}fa$.

Therefore $\overline{D}fa$ = $\overline{B}fa \wedge \vee L_f\overline{C}(f^{-1}fa \cap C)$ $\leq$ $\overline{B}fa \wedge \overline{Y}fa$ = $\overline{Y}fa$, because $\mathsf{Y} \subseteq \mathsf{B}$.

Now, again Y is L_f-regular and hence $\overline{Y}fa \in L_f L_A$ and $\overline{D}fa \leq \overline{Y}fa$ imply, by 3.3.2, $\vee L_f^{-1}\overline{D}fa$ $\leq$ $\vee L_f^{-1}\overline{Y}fa$, as required.

(b): Let F be decreasing. Then $\overline{B}f$ $\leq$ $L_f\overline{A}$. Since $\mathsf{Y} \subseteq \mathsf{B}$, $\overline{Y}f \leq \overline{B}f \leq L_f\overline{A}$. Therefore for any $c \in C$, $L_f\overline{C}c$ = $L_f\overline{A}c \wedge L_f(\vee L_f^{-1}\overline{Y}fc)$ = $L_f\overline{A}c \wedge \overline{Y}fc$ = $\overline{Y}fc = \overline{Y}fa$, because (i) Y is L_f-regular and hence $\overline{Y}fc \in L_Y \subseteq L_f L_A$ and (ii) by 3.3.11(3), $L_f(\vee L_f^{-1}\overline{Y}fc)$ = $\overline{Y}fc$. In particular, $\vee L_f\overline{C}(f^{-1}fa \cap C)$ = $\vee_{c \in f^{-1}fa \cap C} L_f\overline{C}c$ = $\vee_{c \in f^{-1}fa \cap C}\overline{Y}fc$ = $\overline{Y}fa$, implying $\overline{D}fa$ = $\overline{B}fa \wedge \vee L_f\overline{C}(f^{-1}fa \cap C)$ = $\overline{B}fa \wedge \overline{Y}fa$ = $\overline{Y}fa$, because $\mathsf{Y} \subseteq \mathsf{B}$ and hence $\overline{Y} \leq \overline{B}|Y$.

Now clearly $\overline{E}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{Y}fa$ = $\overline{C}a$.

The following example shows that the above proposition is *not* true if Y is *not* L_f-regular:

Example 6.5: Let $\mathsf{F}:\mathsf{A} \to \mathsf{B}$ be defined by: A = $(\{a\},\{(a,1)\},\{0,\alpha,1 \mid 0 < \alpha < 1\})$, B = $(\{b\},\{(b,1)\},\{0,\beta,1 \mid 0 < \beta < 1\})$, $f = \{(a,b)\}$, $L_f = \{(0,0),(\alpha,0),(1,1)\}$.

Let Y = $(\{b\},\{(b,\beta)\},\{0,\beta \mid 0 < \beta\})$.

Then L_Y = $\{0,\beta\}$ $\not\subseteq$ $L_f L_A$ = $\{0,1\}$, implying Y is *not* L_f-regular and $\overline{B}fa$ = 1 = $L_f\overline{A}a$, implying F is preserving.

Let $\mathsf{F}_p^{-1}\mathsf{Y}$ = C. $C = f^{-1}Y$ = $\{a\}$, L_C = $L_f^{-1}L_Y$ = $\{0\}$ and $\overline{C}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{Y}fa$ = $1 \wedge \vee\phi$ = $1 \wedge 0$ = 0.

Let $\mathsf{F}_p\mathsf{C}$ = D. Then D = fC = $\{b\}$, L_D = $(L_f L_C)_{L_B}$ = $\{0\}$ and $\overline{D}b$ = $\overline{B}b \wedge \vee L_f\overline{C}(f^{-1}b \cap C)$ = $1 \wedge 0$ = 0.

Let $\mathsf{F}_p^{-1}\mathsf{D}$ = E. Then E = $f^{-1}D$ = $\{a\}$ = C, L_E = $L_f^{-1}L_D$ = $\{0,\alpha\}$ $\supset$ L_C and so $\mathsf{F}_p^{-1}\mathsf{F}_p\mathsf{F}_p^{-1}\mathsf{Y}$ = E $\neq$ C $\mathsf{F}_p^{-1}\mathsf{Y}$. In fact, also, $\overline{E}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ = $1 \wedge \alpha$ = α $>$ 0 = $\overline{C}a$, implying $\overline{E} \neq \overline{C}$ or $\mathsf{F}_p^{-1}\mathsf{F}_p\mathsf{F}_p^{-1}\mathsf{Y}$ = E $\neq$ C = $\mathsf{F}_p^{-1}\mathsf{Y}$.

Definition 6.6: For any $\mathsf{F}:\mathsf{A} \to \mathsf{B}$ and for any f-subset C of A, C is said to be L_f*-coregular* iff $\overline{B}fC \subseteq L_f L_A$.

Proposition 6.7: For any 0-p $\mathsf{F}:\mathsf{A} \to \mathsf{B}$ and for any L_f-coregular f -subset C of A, we have $\mathsf{F}_*\mathsf{F}_*^{-1}\mathsf{F}_*\mathsf{C}$ = $\mathsf{F}_*\mathsf{C}$ holds whenever $*$ = i or d or p.

Proof: Let FC = D. Then D = fC, L_D = $(L_f L_C)_{L_B}$ and $\overline{D}b$ = $\overline{B}b \wedge \vee L_f\overline{C}(f^{-1}b \cap C)$ for all $b \in D$.

Let $\mathsf{F}^{-1}\mathsf{D}$ = E. Then E = $f^{-1}D$, L_E = $L_f^{-1}L_D$ and $\overline{E}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ for all $a \in E$.

Let $\mathsf{FE} = \mathsf{G}$. Then $G = fE$, $L_G = (L_f L_E)_{L_B}$ and $\overline{G}b$ = $\overline{B}b \wedge \vee L_f\overline{E}(f^{-1}b \cap E)$ for all $b \in G$.

we show that D = G or (1) $D = G$ (2) $L_D = L_G$ and (3) $\overline{D} = \overline{G}$.

(a): G = fE = $ff^{-1}D$ = $ff^{-1}fC$ = fC = D.

(b): By 3.2.3(3), L_C = $[0,\alpha]$ for some $\alpha \in L_A$. By 3.4.3(2), L_D = $(L_f L_C)_{L_B}$ = $(L_f[0,\alpha])_{L_B}$ = $[0, L_f\alpha]$.

By 3.4.6(3), since F and hence L_f is 0-p and $L_f\alpha \in L_f L_A$, L_E = $L_f^{-1}L_D$ = $L_f^{-1}[0, L_f\alpha]$ = $[0, \vee L_f^{-1}L_f\alpha]$.

Again since $L_f\alpha$ $\in$ $L_f L_A$ by 3.4.3(2) and 3.3.11(3), L_G = $(L_f L_E)_{L_B}$ = $(L_f[0, \vee L_f^{-1}L_f\alpha])_{L_B}$ = $[0, L_f(\vee L_f^{-1}L_f\alpha)] = [0, L_f\alpha]$ = L_D.

(c): Let $b \in G$ (= fE = fC = D) be fixed. Then $f^{-1}b \cap C \neq \phi$ and $f^{-1}b \cap E \neq \phi$.

(a) Let F be decreasing. Then $\overline{B}f \leq L_f\overline{A}$. Since $\mathsf{D} \subseteq \mathsf{B}$, $\overline{D} \leq \overline{B}|D$ and hence $\overline{D}f \leq \overline{B}f \leq L_f\overline{A}$.

Since (i) $L_f\overline{C}(f^{-1}b \cap C) \subseteq L_f\overline{C}C \subseteq L_f L_C \subseteq L_f L_A$ (ii) $\overline{B}b \in \overline{B}fC \subseteq L_f L_A$ because C is L_f-coregular and

(iii) $L_f L_A$ is a complete sub lattice, we get that $\overline{D}b$ = $\overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C) \in L_f L_A$. Consequently, by 3.3.11(3), $L_f(\vee L_f^{-1}\overline{D}b)$ = $\overline{D}b$.

Now for all $e \in f^{-1}b \cap E$, $fe = b$ and from the definition of $\overline{E}e$ above, $L_f \overline{E}e$ = $L_f(\overline{A}e \wedge \vee L_f^{-1}\overline{D}fe)$ = $L_f \overline{A}e \wedge L_f(\vee L_f^{-1}\overline{D}fe)$ = $L_f \overline{A}e \wedge \overline{D}fe$ = $\overline{D}fe = \overline{D}b$, where the last but one equality follows because of F being decreasing.

Therefore, $\vee L_f \overline{E}(f^{-1}b \cap E) = \vee_{e \in f^{-1}b \cap E} L_f \overline{E}e$ = $\vee_{e \in f^{-1}b \cap E} \overline{D}fe$ = $\vee_{e \in f^{-1}b \cap E} \overline{D}b$ = $\overline{D}b$.

On the other hand, $\overline{G}b = \overline{B}b \wedge \vee L_f \overline{E}(f^{-1}b \cap E)$ = $\overline{B}b \wedge \overline{D}b$ = $\overline{D}b$, since $D \subseteq B$ and hence $\overline{D} \le \overline{B}\,|\,D$.

(b): Let F be increasing. Then For any increasing f-map, by 5.5.8, $C \subseteq F_*^{-1}F_*C$ for all $C \subseteq A$. So, by 5.5.3, monotonicity of F_* implies D = $F_*C \subseteq F_*F_*^{-1}F_*C$ = G. Hence it is enough to show that $\overline{G} \le \overline{D}$.

For all $e \in f^{-1}b \cap E$, $fe = b$, $fe \in fC(= D = G = fE)$ and as in (a) above, $\overline{D}fe \in L_f L_A$ and $L_f(\vee L_f^{-1}\overline{D}fe) = \overline{D}fe = \overline{D}b$.

Now $\overline{E}e \le \vee L_f^{-1}\overline{D}fe$ for all $e \in f^{-1}b \cap E$, implying $L_f \overline{E}e \le L_f(\vee L_f^{-1}\overline{D}fe)$ = $\overline{D}fe$ = $\overline{D}b$ and $\overline{G}b$ = $\overline{B}b \wedge \vee L_f \overline{E}(f^{-1}b \cap E)$ $\le$ $\vee L_f \overline{E}(f^{-1}b \cap E)$ = $\vee_{e \in f^{-1}b \cap E} L_f \overline{E}e$ $\le$ $\overline{D}b$ or $\overline{G} \le \overline{D}$.

The following example shows that the above proposition is *not* true if C isnot L_f -coregular but F is 0-p.

Example 6.8: Let $F: A \to B$ be defined by: A = $(\{a\}, \{(a,1)\}, \{0,1 \,|\, 0 < 1\})$, B = $(\{b\}, \{(b,\alpha)\}, \{0,\alpha,1 \,|\, 0 < \alpha < 1\})$ and F = $(\{(a,b)\}, \{(0,0), (1,1)\})$.

Then α = $\overline{B}b$ = $\overline{B}fa$ $\le$ $L_f \overline{A}a$ = $L_f 1$ = 1, implying F is 0-p and decreasing, and $\overline{B}fC \not\subseteq L_f L_A$, implying C is *not* L_f -coregular.

Letting $C = A$ and $D = F_dC$, we get that $\overline{D}b$ = $\overline{B}b$ $\wedge$ $\vee L_f \overline{C}(f^{-1}b \cap C) = \alpha \wedge 1 = \alpha$.

Letting $E = F_d^{-1}D$, we get that $\overline{E}a$ = $\overline{A}a$ $\wedge$ $\vee L_f^{-1}\overline{D}fa$ = $1 \wedge \vee \phi$ = $1 \wedge 0$ = 0.

Letting $G = F_dE$, we get that $\overline{G}b$ = $\overline{B}b$ $\wedge$ $\vee L_f \overline{E}(f^{-1}b \cap E)$ = $\overline{B}b$ $\wedge$ 0 = 0, implying $F_dF_d^{-1}F_dC \not\subseteq F_dC$.

Proposition 6.9: Forany increasing f-map $F: A \to B$ and for any pair of f-subsets C of A and D of B, $FC \subseteq D$ implies $C \subseteq F^{-1}D$ whenever D is L_f -regular.

Proof: Let F_iC = E. Then E = fC, L_E = $(L_f L_C)_{L_B}$ and $\overline{E}b$ = $\overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C)$ for all $b \in E$.

Let $F_i^{-1}D$ = G. Then G = $f^{-1}D$, L_G = $L_f^{-1}L_D$ and $\overline{G}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ for all $a \in G$.

Since $E \subseteq D$, $E \subseteq D$, L_E is a complete ideal of L_D and $\overline{E} \le \overline{D}\,|\,E$.

We show that $C \subseteq G$ or (1) $C \subseteq G$ (2) L_C is a complete ideal of L_G and (3) $\overline{C} \le \overline{G}\,|\,C$.

(a): Since $fC \subseteq D$ iff $C \subseteq f^{-1}D$, $C \subseteq f^{-1}D$ = G.

(b): Since L_E is a complete ideal of L_D, $L_f L_C \subseteq (L_f L_C)_{L_B}$ = L_E $\subseteq$ L_D. So, $L_C \subseteq L_f^{-1}L_D$ = L_G.

Since L_G and L_C are complete ideals of L_A, it follows from $L_C \subseteq L_G$ that L_C is a complete ideal of L_G.

(c): Let $a \in C$ be fixed. Then $fa \in fC = E$. $\overline{G}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$. Since $\overline{A}a \ge \overline{C}a$ to show $\overline{C} \le \overline{G}\,|\,C$, it is enough to show that $\vee L_f^{-1}\overline{D}fa \ge \overline{C}a$.

Since (i) $a \in f^{-1}fa \cap C$, $L_f \overline{C}a \le \vee L_f \overline{C}(f^{-1}fa \cap C)$ and (ii) $\overline{E} \le \overline{D}\,|\,E$, we get that $\overline{B}fa \wedge L_f \overline{C}a \le \overline{B}fa \wedge \vee L_f \overline{C}(f^{-1}fa \cap C)$ = $\overline{E}fa$ $\le$ $\overline{D}fa$.

Since $C \subseteq A$ and F is increasing, $L_f \overline{C}a \le L_f \overline{A}a \le \overline{B}fa$ which implies $L_f \overline{C}a$ = $\overline{B}fa \wedge L_f \overline{C}a \le \overline{D}fa$, from the above.

Since (i) $\overline{D}fa \in L_D \subseteq L_f L_A$ as D is L_f -regular (ii) $L_f \overline{C}a \le \overline{D}fa$, by 3.3.2, $\overline{C}a$ $\le$ $\vee L_f^{-1}L_f \overline{C}a$ $\le$ $\vee L_f^{-1}\overline{D}fa$ as required.

The following example shows that the above proposition is *not* true if D is *not* L_f -regular but F is increasing:

Example 6.10: Let $F: A \to B$ be defined by: A = $(\{a\}, \{(a,1)\}, \{0,\alpha,1 \,|\, 0 < \alpha < 1\})$, B = $(\{b\}, \{(b,1)\}, \{0,\beta,1 \,|\, 0 < \beta < 1\})$, $f = \{(a,b)\}$ and $L_f = \{(0,0), (\alpha,0), (1,1)\}$. Let C =

$(\{a\},\{(a,\alpha)\},\{0,\alpha \mid 0<\alpha\})$ and D = $(\{b\},\{(b,\beta)\},\{0,\beta \mid 0<\beta\})$.

Then $L_D = \{0,\beta\} \not\subseteq \{0,1\} = L_f L_A$, implying D is *not* L_f-regular, $\overline{B}fa = 1 = L_f \overline{A}a$, implies F is preserving.

Let $F_p C = E$. Then $E = fC = \{b\} = D$, $L_E = (L_f L_C)_{L_B} = \{0\} \subseteq L_D = \{0,\beta\}$ and $\overline{E}b = \overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C) = 1 \wedge 0 = 0 \le \overline{D}b = \beta$, implying $FC \subseteq D$.

Let $F_p^{-1}D = G$. Then $G = f^{-1}D = \{a\} = C$, $L_G = L_f^{-1}L_D = \{0,\alpha\} = L_C$ but $\overline{G}a = \overline{A}a \wedge \vee L_f^{-1}\overline{D}fa = 1 \wedge \vee\phi = 1 \wedge 0 = 0 < \alpha = \overline{C}a$, implying $\overline{C}a \not\le \overline{G}a$ or $C \not\subseteq F^{-1}D$.

The following example showsthat the above proposition is *not* true if F is decreasing but D is L_f-regular:

Example 6.11: Let $F: A \to B$ be defined by: A = $(\{a\},\{(a,1)\},\{0,\alpha,\beta,\gamma,1 \mid 0<\alpha<\beta,\gamma<1;\beta \parallel \gamma\})$, B = $(\{b\},\{(b,\beta)\},\{0,\alpha,\beta,\gamma,1 \mid 0<\alpha<\beta,\gamma<1;\beta \parallel \gamma\})$, $f = \{(a,b)\}$ and $L_f = \{(0,0),(\alpha,\alpha),(\beta,\beta),(\gamma,\gamma),(1,1)\}$. *Let* C = $(\{a\},\{(a,\gamma)\},L_A)$ *and* D = $(\{b\},\{(b,\alpha)\},L_B)$.

Then $\overline{B}fa = \beta < 1 = L_f \overline{A}a$, implying F is decreasing and $L_D = L_B = L_f L_A$, implying D is L_f-regular.

Let $F_d C = E$. Then $E = fC = \{b\} = D$, $L_E = (L_f L_C)_{L_B} = (L_f L_A)_{L_B} = L_B = L_D$ and $\overline{E}b = \overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C) = \beta \wedge \gamma = \alpha = \overline{D}b$, implying $F_d C = E = D$.

Let $F_d^{-1}D = G$. Then $G = f^{-1}D = \{a\} = C$, $L_G = L_f^{-1}L_D = L_f^{-1}L_B = L_A$ and $\overline{G}a = \overline{A}a \wedge \vee L_f^{-1}\overline{D}fa = 1 \wedge \alpha = \alpha \not\ge \gamma = \overline{C}a$, implying $C \not\subseteq G = F^{-1}D$.

Proposition 6.12: For any f-map $F: A \to B$ and for any pair of f-subsets C of A and D of B, $C \subseteq F^{-1}D$ implies $FC \subseteq D$, whenever F is 0-p or D is L_f-regular.

***Proof*:** Let $FC = E$. Then $E = fC$, $L_E = (L_f L_C)_{L_B}$ and $\overline{E}b = \overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C)$ for all $b \in E$.

Let $F^{-1}D = G$. Then $G = f^{-1}D$, $L_G = L_f^{-1}L_D$ and $\overline{G}a = \overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ for all $a \in G$.

Since $C \subseteq G$, we have $C \subseteq G$, L_C is a complete ideal of L_G and $\overline{C} \le \overline{G}|C$.

We show that $E \subseteq D$ or (1) $E \subseteq D$ (2) L_E is a complete ideal of L_D and (3) $\overline{E} \le \overline{D}|E$.

(a): $C \subseteq G = f^{-1}D$ implies $fC \subseteq D$ which implies $E \subseteq D$.

(b): Since $L_C \subseteq L_G = L_f^{-1}L_D$, $L_f L_C \subseteq L_D$ and L_D is a complete ideal of L_B implies $L_E = (L_f L_C)_{L_B} \subseteq L_D$. Since L_E and L_D are complete ideals of L_B such that $L_E \subseteq L_D$, we get that L_E is a complete ideal of L_D.

(3): Let $b \in E = FC$ be fixed. For any $a \in f^{-1}b \cap C$, $a \in C$ and $b = fa \in fC = D$.

Since (i) F and hence L_f is 0-p,by 3.3.11(4), $L_f(\vee L_f^{-1}\overline{D}fa) \le \overline{D}fa$ or

(ii) D is L_f-regular, so $L_D \subseteq L_f L_A$ and hence $\overline{D}fa \in L_D \subseteq L_f L_A$, by 3.3.11(3), $L_f(\vee L_f^{-1}\overline{D}fa) = \overline{D}fa$.

But as $C \subseteq G$, $\overline{C} \le \overline{G}|C$ and this implies $L_f\overline{C} \le L_f\overline{G}$ and hence from the above, $L_f\overline{C}a \le L_f\overline{G}a = L_f(\overline{A}a \wedge \vee L_f^{-1}\overline{D}fa) = L_f\overline{A}a \wedge L_f(\vee L_f^{-1}\overline{D}fa) \le L_f\overline{A}a \wedge \overline{D}fa \le \overline{D}fa = \overline{D}b$ for all $a \in f^{-1}b \cap C$, implying $\vee L_f \overline{C}(f^{-1}b \cap C) \le \overline{D}b$ and $\overline{E}b = \overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap C) \le \vee L_f \overline{C}(f^{-1}b \cap C) \le \overline{D}b$, implying $\overline{E} \le \overline{D}$ or $FC = E \subseteq D$.

The following example shows that the above proposition is *not* true if both F is *not* 0-p and D is *not* L_f-regular:

Example 6.13: Let $F: A \to B$ be defined by: A = $(\{a\},\{(a,1)\},\{0,\alpha,1 \mid 0<\alpha<1\})$, B = $(\{b\},\{(b,1)\},\{0,\beta,\gamma,1 \mid 0<\beta<\gamma<1\})$, $f = \{(a,b)\}$ and $L_f = \{(0,\gamma),(\alpha,\gamma),(1,1)\}$.

Let C = $(\{a\},\{(a,0)\},\{0,\alpha \mid 0<\alpha\})$, D = $(\{b\},\{(b,\beta)\},\{0,\beta,\gamma \mid 0<\beta<\gamma\})$.

Then F is *not* 0-p, $L_D = \{0,\beta,\gamma\} \not\subseteq \{\gamma,1\} = L_f L_A$ implying, D is *not* L_f-regular and $\overline{B}fa = 1 = L_f\overline{A}a$ implying F is preserving.

Let $F_p^{-1}D = G$. Then $G = f^{-1}D = \{a\} = C$, L_G =

$L_f^{-1}L_D$ = $\{0,\alpha\}$ = L_C and $\overline{G}a$ = $\overline{A}a \wedge \vee L_f^{-1}\overline{D}fa$ = $1 \wedge \vee \phi$ = $1 \wedge 0$ = 0 = $\overline{C}a$ or $\mathsf{C} = \mathsf{G} = \mathsf{F}^{-1}\mathsf{D}$.

Let $\mathsf{F}_p\mathsf{C}$ = E. Then E = fC = $\{b\}$ = D, L_E = $(L_f L_C)_{L_B}$ = $\{0,\beta,\gamma\}$ = L_D and $\overline{E}b$ = $\overline{B}b \wedge \vee L_f \overline{C}(f^{-1}b \cap c)$ = $1 \wedge \gamma$ = γ > β = $\overline{D}b$ or FC = E $\supset$ D or FC $\not\subseteq$ D.

Lemma 6.14: For any f-map $\mathsf{F}:\mathsf{X} \to \mathsf{Y}$ and for any f-subset A of X, A = Φ iff FA = Φ.

Proof: $(\Rightarrow)$: A = Φ implies $A = \phi$, $L_A = \phi$ and $\overline{A} = \phi$. FA = C implies $C = fA = f\phi = \phi$, L_C = $(L_f L_A)_{L_B} = \phi$ and $\overline{C} \subseteq C \times L_C$ = ϕ, implying FA = $\mathsf{C} = \Phi$.

$(\Leftarrow)$: FA = C = Φ implies, C = $fA = \phi$ which implies $A = \phi$, since $fA = \phi$ iff $A = \phi$; $L_f L_A \subseteq (L_f L_A)_{L_B} = L_C = \phi$, implying $L_f L_A = \phi$ which implies $L_A = \phi$ and $\overline{A} \subseteq A \times L_A = \phi \times \phi$ implies $\overline{A} = \phi$ or A = Φ.

Corollary 6.15: *For any 1-p f-map* $\mathsf{F}:\mathsf{X} \to \mathsf{Y}$ *and forany nonempty family* $(\mathsf{A}_i)_{i\in I}$ *of f-subsets of* X, $\cap_{i\in I} \mathsf{FA}_i$ = Φ *implies* $\cap_{i\in I} A_i$ = Φ.

Proof: It follows from the above Lemma and 5.5.21.

Lemma 6.16: For any f-map $\mathsf{F}:\mathsf{X} \to \mathsf{Y}$, $\mathsf{F}^{-1}\Phi$ = Φ.

Proof: $\mathsf{F}^{-1}\phi = \mathsf{C}$ implies C = $f^{-1}\phi = \phi$, $L_C = L_f^{-1}\phi = \phi$ and $\overline{C} \subseteq C \times L_C = \phi \times \phi = \phi$, implying $\mathsf{F}^{-1}\phi$ = C = $(\phi,\phi,\phi) = \phi$.

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