ISSN No. 0976-5697
Volume 5, No. 6, July-August 2014
International Journal of Advanced Research in Computer Science
RESEARCH PAPER
Available Online at www.ijarcs.info

Stabilization of Free Rigid Body Motion Stereo 3D Simulation through Invariants

Svetoslav Zabunov
Space Research and Technology Institute at the Bulgarian Academy of Sciences
Sofia, Bulgaria

Petar Getsov
Space Research and Technology Institute at the Bulgarian Academy of Sciences
Sofia, Bulgaria

Maya Gaydarova
Methodology of Physics Department
Physics Faculty at Sofia University
Sofia, Bulgaria

Abstract: A good spatial perception of the investigated physical phenomenon is important for obtaining successful outcome of the research process. Stereoscopic 3D simulations in a full-fledged online environment are applicable in scientific research of satellite and unmanned aerial vehicle motion. Simulating complex mechanical problems for scientific and experimental tasks requires not only precision to certain degree, but also correct and consistent representation of the physical laws inherent to the simulated formalism. To attain this aim, a genuine stabilization approach is needed.

The current paper describes an environment for simulation of free rigid body motion while conserving certain invariants in order not to violate the underlying formalism of the Newtonian classical mechanics even under the inevitable errors inherent to the numerical integration. The dynamic stabilization through invariants of the free rigid body motion simulation is depicted. The simulation, subject of the current paper, may be observed on http://ialms.net/sim/ web address.

Keywords: 3D stereoscopic simulations of rigid body motion, Unmanned aerial vehicles (UAV) simulation, Satellite simulation.

I. INTRODAUCTION

The current paper presents the stabilization of free rigid body motion simulation implemented in stereoscopic 3-dimensional (3D) online environment. Such a simulation represents great benefit for the scientific research society in the field of unmanned aerial vehicle (UAV) and satellite motion investigation. The stabilization of free rigid body motion simulation was realized in pursuit of validity of the simulated process in relation to the examined laws of physics, corresponding to the underlying Newtonian classical mechanics formalism.

The simulation, described in the current paper, can be viewed on http://ialms.net/sim/ web address (Fig. 1).

After a conducted survey of existing online 3D stereoscopic simulations of mechanical phenomena it was discovered that such simulations do not exist. At the same time, it is clear that an approach of this kind would generate a substantial benefit for the modern physics research avenue. The inevitable error accumulation due to numerical simulation integration steps is canalized in such ways as to resemble the perturbation errors found in real laboratory experiments, while laws of mechanics are kept intact and all conserved quantities preserved.

II. SIMULATION STABILIZATION BACKGROUND

First of all, a definition of stabilization is required. In the rigid body motion, and especially in the free rigid body motion, the most common meaning of the term stabilization is to keep a free moving rigid body rotating along a given axis and preventing it from deviation of the axis of rotation. This approach is common in aeronautics and space satellites where stabilization is the process of applying torques (reactive jets) to the rigid body (satellite or spacecraft) in order to keep it rotating along the desired axis. For this purpose numerous models and approaches have been developed, such as [1-3]. In the mentioned materials the Serret-Andoyer formalism is used to model the free rigid body rotation and reduce the motion variables to one (one degree of freedom). This formalism is based on the Euler reduction parameters of the rotation matrix. The latter is the general component, describing the orientation of the rigid body in every moment of time. The Euler angels approach is in contrast to the current project, where Euler variables are avoided in favor of the rotation quaternion reduction of the rotation matrix, due to quaternions inherent performance advantage over other reductions of the rotation matrix and the absence of certain limitations, such as gimbal lock, innate to the Euler parameters model.

The other common meaning of stabilization is presented by the current material. It is to stabilize a simulation of a mechanical process around certain principles, laws and invariants, against the inevitable calculation error accumulation. Such common methods are described in [5, 6]. Interactive simulation of mechanical phenomena presumes error accumulation in the state variables describing the simulated rigid bodies and, as a consequence, the simulation process acquires error perturbation. This perturbation consists of deviation from the real state. The most common real-time simulations of rigid body motion, such as computer games, require building a sense of reality in the observer. Other simulations, such as scientific simulations, demand certain accuracy to be always met. But nevertheless the accuracy maintained, the errors are always accumulated and as the simulation prolongs its period the errors deviate further and further the simulated state from the real state. The simulation algorithm has a choice how to manage the accumulated errors and the best to do this is keep physics laws intact even when the accuracy is essentially lost - after a very long time of simulation and the unavoidable enormous error accumulation that would follow. As a result, certain qualitative characteristics of the simulated process should be preserved. In such a scenario, error perturbations should not degrade the principle laws in the simulated mechanical setup.

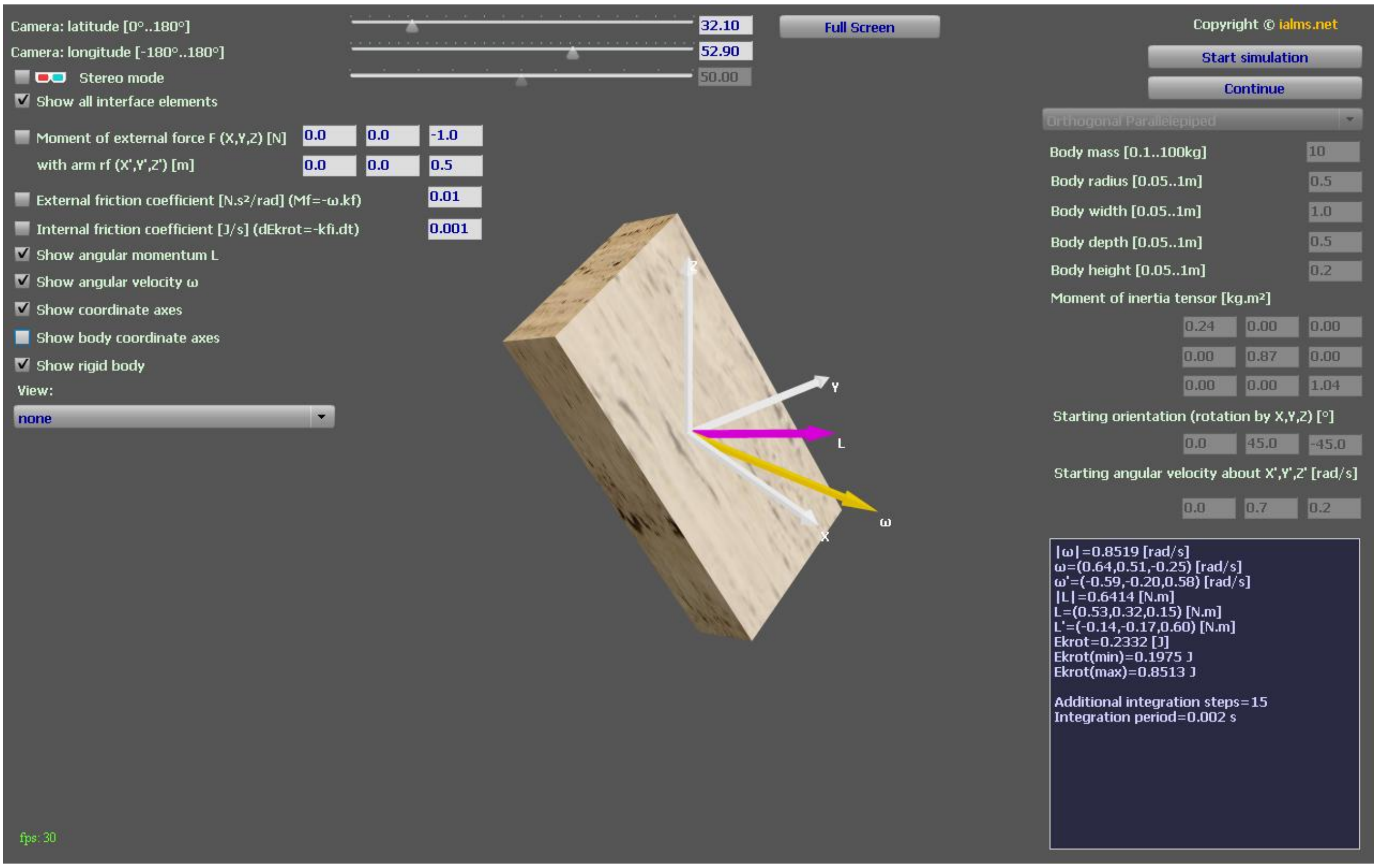

Figure 1. Simulation of rigid body motion. The space reference frame basis is shown along with the angular velocity (orange color) and angular momentum (magenta color) vectors.

Error accumulation should be canalized to state variables errors that are the same in nature as the perturbations observed in real world laboratory experiments coming from unknown outer perturbing factors.

In the case of free rigid body motion, two constraints can be utilized to achieve better simulation. These are two variables of the rigid body state, which are, under these conditions, invariant.

III. STABILIZATION THROUGH INVARIANT CONSTRUCTIONS

The description of the stabilization approach would require a concise overview of the basic terms and formalism utilized in order to clarify the presentation. When talking about rigid body motion, a non-inertial reference frame needs to be defined. This reference frame is connected to the body and is called the body frame. The rigid body does not move, nor rotate in respect to the body frame. Thus the motion of the body equals the motion of the body frame in respect to the inertial space frame. This motion is linear and rotational with six degrees of freedom [4, 7]. Table 1 shows the variables, describing the two motions (linear and rotational) of the rigid body. All variables are compared one by one. Such tables are common in theoretical mechanics textbooks, but very often they miss fundamental variables and relationships such as rotational position (orientation). Table 1 is generalized and utilizes the matrix presentation of the state variables, describing the rigid body state.

A brief overview of notations and equations used in the current paper follow. Vectors are preferably presented in a matrix form, either as row-matrix or column-matrix, as follows:

$$\vec{b} \Leftrightarrow \mathbf{b} = \begin{bmatrix} b_x & b_y & b_z \end{bmatrix}$$

Matrices are denoted with bold characters. Transposed matrix of matrix $\mathbf{R}$ is denoted with $\mathbf{R}^{\sim}$ and the derivative in respect to time is expressed with the dot notation $\dot{\mathbf{R}}$. In this paper, the anti-symmetric matrix of a vector will be used extensively. If vector $\vec{b}$ is given in matrix form $\mathbf{b}$, then its anti-symmetric matrix is denoted with $\mathbf{b}^*$ and is equal to:

$$\mathbf{b}^* = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}$$

Matrix $\mathbf{b}^*$ has three degrees of freedom and is isomorphic to the 3D vector $\mathbf{b}$.

When vectors are presented in matrix form, the equivalence between vector product and the multiplication with anti-symmetric matrix of a vector is as follows:

$$\vec{a} \times \vec{b} \Leftrightarrow \mathbf{ab}^* = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}$$

The outer product of two vectors, presented in matrix form, should also be recalled:

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_x & b_y & b_z \end{bmatrix} = \mathbf{a}^{\sim}\mathbf{b}$$

Note that both matrix products $\mathbf{b} \otimes \mathbf{b}$ and $\mathbf{b}^{*2}$ yield symmetric matrices as follows:

$$\mathbf{b} \otimes \mathbf{b} = \begin{bmatrix} b_x^2 & b_x b_y & b_x b_z \\ b_x b_y & b_y^2 & b_y b_z \\ b_x b_z & b_y b_z & b_z^2 \end{bmatrix}$$

$$\mathbf{b}^{*2} = \begin{bmatrix} -b_z^2 - b_y^2 & b_x b_y & b_x b_z \\ b_x b_y & -b_x^2 - b_z^2 & b_y b_z \\ b_x b_z & b_y b_z & -b_x^2 - b_y^2 \end{bmatrix} =$$

$$\mathbf{b} \otimes \mathbf{b} - \mathbf{bb}^{\sim}\mathbf{1} = \mathbf{b}^{\sim}\mathbf{b} - \mathbf{bb}^{\sim}\mathbf{1} = \mathbf{b} \otimes \mathbf{b} - b^2\mathbf{1}$$

The latter equation is frequently utilized. Matrix $\mathbf{1}$ is the 3x3 identity matrix.

The variables, describing the rigid body linear and rotational motion are rendered systematic in table 1.

From the relation between the angular momentum and angular velocity follows:

$$\mathbf{L} = \boldsymbol{\omega}\mathbf{I} \Rightarrow \mathbf{LR}^{\sim} = \boldsymbol{\omega}\mathbf{R}^{\sim}\mathbf{RIR}^{\sim} \Rightarrow \mathbf{L}' = \boldsymbol{\omega}'\mathbf{I}'$$

Rigid body dynamic parameters for linear and rotational motion

Parameter	Linear motion	Angular (rotational) motion
Position	Linear position is presented by the radius-vector of the center of mass $\mathbf{r}_c$. All linear variables are with respect to the space reference frame, while the origin of the body reference frame coincides with the center of mass $\mathbf{r}_c = \overrightarrow{OO'}$.	Angular position or orientation is expressed by the rotation matrix $\mathbf{R}$ or any of its reduction derivatives, such as Euler angles, rotation quaternion, etc. The rotation matrix connects the space and the body reference frames by the following operator: $(...)'\mathbf{R} = (...) \Rightarrow (...)\mathbf{R}^{\sim} = (...)'$ Prime-variables are defined in the body reference frame.
Velocity	Linear velocity vector of the center of mass $\mathbf{v}_c = \dot{\mathbf{r}}_c$.	Angular velocity vector $\boldsymbol{\omega}$, where $\boldsymbol{\omega}^* = \dot{\mathbf{R}}^{\sim}\mathbf{R}$ and $\boldsymbol{\omega} = \begin{pmatrix} \omega_{32}^* & \omega_{13}^* & \omega_{21}^* \end{pmatrix}$. In the body reference frame the angular velocity is $\boldsymbol{\omega}' = \boldsymbol{\omega}\mathbf{R}^{\sim}$.
Acceleration	Linear acceleration vector of the center of mass $\mathbf{a}_c = \ddot{\mathbf{r}}_c$.	Angular acceleration vector $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$ or $\boldsymbol{\varepsilon}^* = \ddot{\mathbf{R}}^{\sim}\mathbf{R} + \dot{\mathbf{R}}^{\sim}\dot{\mathbf{R}}$ and $\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{32}^* & \varepsilon_{13}^* & \varepsilon_{21}^* \end{pmatrix}$. In the body reference frame the angular acceleration is $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\mathbf{R}^{\sim}$.
Inertia	Total mass of the rigid body $m = \iiint_V \rho(\mathbf{r})dV =$ $\iiint_V \rho(x, y, z)dxdydz$ Note: $\mathbf{r} = \begin{pmatrix} x & y & z \end{pmatrix}$.	Moment of inertia tensor in the body reference frame $\mathbf{I}' = \begin{bmatrix} I'_{xx} & I'_{xy} & I'_{xz} \\ I'_{xy} & I'_{yy} & I'_{yz} \\ I'_{xz} & I'_{yz} & I'_{zz} \end{bmatrix} = \iiint_{V'} (r'^2\mathbf{1} - \mathbf{r}' \otimes \mathbf{r}')dm =$ $= -\iiint_{V'} \mathbf{r}'^{*2}\rho(\mathbf{r}')dV' = -\iiint_{V'} \mathbf{r}'^{*2}\rho(x', y', z')dx'dy'dz'$

		$\mathbf{r}'^{*2}$ is a symmetric matrix and its integral is also a symmetric matrix. Hence, tensor $\mathbf{I}'$ is a symmetric matrix and has only six degrees of freedom. In the space reference frame, the moment of inertia tensor is $\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} = \mathbf{R}^{\sim}\mathbf{I}'\mathbf{R}$. Here $\mathbf{I} = \mathbf{R}^{\sim}\mathbf{I}'\mathbf{R}$ and $\mathbf{I}' = \mathbf{R}\mathbf{I}\mathbf{R}^{\sim}$ is the well-known similarity transformation or matrix rotation. Note that, while $\mathbf{I}'$ is constant, $\mathbf{I}$ depends on the current body orientation. It is also obvious that $\mathbf{I} = \mathbf{R}^{\sim}\mathbf{I}'\mathbf{R} = -\iiint_{V'} \mathbf{R}^{\sim}\mathbf{r}'^{*}\mathbf{R}\mathbf{R}^{\sim}\mathbf{r}'^{*}\mathbf{R}\rho(\mathbf{r}')dV' =$ $= -\iiint_{V} \mathbf{r}^{*2}\rho(\mathbf{r})dV$ Matrix $\mathbf{I}$ is also symmetric, because rotation preserves symmetry and anti-symmetry. The moment of inertia tensor $\mathbf{I}$ is a tensor of second rank that relates vector $\mathbf{L}$ to vector $\boldsymbol{\omega}$ (see below). $\mathbf{I}$ is equivalent to a 3x3 matrix.
Momentum	Linear momentum of the center of mass $\mathbf{p}_c = m\mathbf{v}_c$.	Angular momentum. Its differential form is $d\mathbf{L}(\mathbf{r}) = \mathbf{r}d\mathbf{p}(\mathbf{r})^{*} = \mathbf{r}(\mathbf{v}(\mathbf{r})dm)^{*} = \mathbf{r}(\boldsymbol{\omega}\mathbf{r}^{*}dm)^{*} = -\boldsymbol{\omega}\mathbf{r}^{*2}dm$ and its integral form is $\mathbf{L} = \iiint_V d\mathbf{L}(\mathbf{r}) = -\iiint_V \boldsymbol{\omega}\mathbf{r}^{*2}dm = -\boldsymbol{\omega}\iiint_V \mathbf{r}^{*2}\rho(\mathbf{r})dV = \boldsymbol{\omega}\mathbf{I}$. In the body reference frame the angular momentum is $\mathbf{L}' = \mathbf{L}\mathbf{R}^{\sim}$.
Force	Force $\sum \mathbf{F}_{ext} = m\mathbf{a}_c = m\dot{\mathbf{v}}_c = \dot{\mathbf{p}}_c$. The sum of all external forces, applied to the rigid body, changes the linear momentum of the center of mass in respect to time as follows: $\dot{\mathbf{p}}_c = \sum \mathbf{F}_{ext}$ or $\Delta\mathbf{p}_c = \int_0^{\Delta t} \sum \mathbf{F}_{ext} dt$.	Torque (moment of force) $\boldsymbol{\tau} = \mathbf{r}\mathbf{F}^{*}$. The sum of all external torques, applied to the rigid body, changes the angular momentum in respect to time as follows: $\dot{\mathbf{L}} = \sum \boldsymbol{\tau}_{ext} = \sum \mathbf{r}\mathbf{F}^{*}_{ext}$ or $\Delta\mathbf{L} = \int_0^{\Delta t} \sum \boldsymbol{\tau}_{ext} dt = \int_0^{\Delta t} \sum \mathbf{r}\mathbf{F}^{*}_{ext} dt$. In the body reference frame the torque is $\boldsymbol{\tau}' = \boldsymbol{\tau}\mathbf{R}^{\sim}$.
Kinetic energy	Kinetic energy of the linear motion $E_K = \frac{mv_c^2}{2}$.	Kinetic energy of the rotational motion $E_{Krot} = \iiint_V \frac{(dm)(\mathbf{v}_{rot}(\mathbf{r}))^2}{2} = \frac{1}{2}\iiint_V \boldsymbol{\omega}\mathbf{r}^{*}(\boldsymbol{\omega}\mathbf{r}^{*})^{\sim}\rho(\mathbf{r})dV =$ $= -\frac{1}{2}\iiint_V \boldsymbol{\omega}\mathbf{r}^{*2}\boldsymbol{\omega}^{\sim}\rho(\mathbf{r})dV = \frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\mathbf{n}\mathbf{I}\mathbf{n}^{\sim}\omega^2}{2} = \frac{I\omega^2}{2}$ Note: vector $\mathbf{n} = \frac{\boldsymbol{\omega}}{\omega}$ and $I = \mathbf{n}\mathbf{I}\mathbf{n}^{\sim}$.

The first invariant in free rigid body motion is the angular momentum vector $\mathbf{L}$. It allows the simulator to store the angular momentum instead of angular velocity in the array of variables, describing the rigid body state. Thus accumulation of errors in the first order time derivative of the rotational position of the body is avoided. On each integration step the angular velocity vector $\boldsymbol{\omega}$ is derived from the angular momentum $\mathbf{L}$ as follows:

$$\boldsymbol{\omega} = \mathbf{L}\mathbf{I}^{-1}, \text{ and } \boldsymbol{\omega}' = \mathbf{L}'\mathbf{I}'^{-1} \qquad (1)$$

If the body reference frame is chosen along the principal axis of inertia, the moment of inertia tensor transforms to a diagonal matrix $\mathbf{I}' = \begin{bmatrix} I'_{xx} & 0 & 0 \\ 0 & I'_{yy} & 0 \\ 0 & 0 & I'_{zz} \end{bmatrix} = const$. Its inverse is also diagonal and has the simple form of $\mathbf{I}'^{-1} = \begin{bmatrix} I'^{-1}_{xx} & 0 & 0 \\ 0 & I'^{-1}_{yy} & 0 \\ 0 & 0 & I'^{-1}_{zz} \end{bmatrix} = \begin{bmatrix} 1/I'_{xx} & 0 & 0 \\ 0 & 1/I'_{yy} & 0 \\ 0 & 0 & 1/I'_{zz} \end{bmatrix} = const$.

The second invariant in free rigid body motion is the kinetic energy of rotational motion E_{Krot}. This conservation parameter leads to the following constraint of movement:

$$E_{Krot} = \frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\boldsymbol{\omega}\mathbf{R}^{\sim}\mathbf{R}\mathbf{I}\mathbf{R}^{\sim}\mathbf{R}\boldsymbol{\omega}^{\sim}}{2} = \frac{\boldsymbol{\omega}'\mathbf{I}'(\boldsymbol{\omega}\mathbf{R}^{\sim})^{\sim}}{2} = \frac{\boldsymbol{\omega}'\mathbf{I}'\boldsymbol{\omega}'^{\sim}}{2} = const \tag{2a}$$

Analogously, the above equation can be transformed in respect to the angular momentum:

$$E_{Krot} = \frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\mathbf{L}\mathbf{I}^{-1}\mathbf{I}(\mathbf{L}\mathbf{I}^{-1})^{\sim}}{2} = \frac{\mathbf{L}\mathbf{I}^{-1}\mathbf{I}\mathbf{I}^{-1\sim}\mathbf{L}^{\sim}}{2} = \frac{\mathbf{L}\mathbf{I}^{-1}\mathbf{L}^{\sim}}{2} = \frac{\mathbf{L}'\mathbf{I}'^{-1}\mathbf{L}'^{\sim}}{2} = const \tag{2b}$$

Evolving these two equations by the vector components and the principal moments of inertia gives:

$$\boldsymbol{\omega}'\mathbf{I}'\boldsymbol{\omega}'^{\sim} = \omega'^2_x I'_{xx} + \omega'^2_y I'_{yy} + \omega'^2_z I'_{zz} = 2E_{Krot} = const \tag{3a}$$

$$\mathbf{L}'\mathbf{I}'^{-1}\mathbf{L}'^{\sim} = L'^2_x I'^{-1}_{xx} + L'^2_y I'^{-1}_{yy} + L'^2_z I'^{-1}_{zz} = 2E_{Krot} = const \tag{3b}$$

These are the equations of two ellipsoids which are static (invariant) in the body reference frame. These two ellipsoids depend solely on the inertial properties of the rigid body and will be denoted with $\varepsilon_{\boldsymbol{\omega}}$ and $\varepsilon_{\mathbf{L}}$ respectively. If the rotational kinetic energy is to stay constant, both vectors $\boldsymbol{\omega}'$ and $\mathbf{L}'$ are constrained to point on the surface of the two ellipsoids defined by equations (3a) and (3b) respectively. If either of vectors $\boldsymbol{\omega}'$ or $\mathbf{L}'$ points outside of its constraint ellipsoid, E_{Krot} will increase. Analogously, if either vector points inside its constraint ellipsoid, E_{Krot} will decrease.

By implementing the $\mathbf{L}$ constraint in the E_{Krot} constraint, one observes that:

$$E_{Krot} = \frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\mathbf{L}\mathbf{I}^{-1}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\mathbf{L}\boldsymbol{\omega}^{\sim}}{2} = \frac{\boldsymbol{\omega}\mathbf{L}^{\sim}}{2} = const \Rightarrow \boldsymbol{\omega}\mathbf{L}^{\sim} = 2E_{Krot} = const \tag{4}$$

In other words, $\boldsymbol{\omega}\mathbf{L}^{\sim} = \omega L\cos\alpha = 2E_{Krot}$, but $\mathbf{L} = const \Rightarrow \omega\cos\alpha = \frac{2E_{Krot}}{L} = const$. The projection vector $\boldsymbol{\omega}_{\mathbf{L}}$ of vector $\boldsymbol{\omega}$ in the direction of vector $\mathbf{L}$ is always the same (Fig. 2):

$$\boldsymbol{\omega}_{\mathbf{L}} = \frac{\boldsymbol{\omega}\mathbf{L}^{\sim}\mathbf{L}}{|\mathbf{L}|^2} = \frac{\omega L\cos\alpha}{L^2}\mathbf{L} = \frac{\omega\cos\alpha}{L}\mathbf{L} = \omega\cos\alpha\,\mathbf{n}_{\mathbf{L}} = \frac{2E_{Krot}}{L}\mathbf{n}_{\mathbf{L}}$$

Here, vector $\mathbf{n}_{\mathbf{L}} = \frac{\mathbf{L}}{L} = \mathbf{n}_{\gamma}$ is the normal vector of an invariable plane γ.

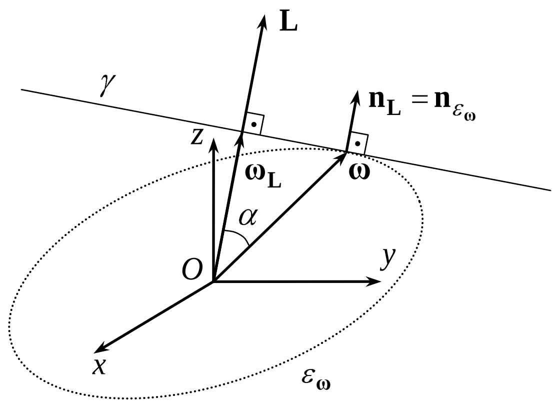

Figure 2. Poinsot construction in the space reference frame.

So, in what boundaries could vector $\boldsymbol{\omega}$ vary? Obviously, $\boldsymbol{\omega}$ could point anywhere in the invariable plane γ, defined by vector $\mathbf{L}$ and E_{Krot}. This invariable plane is specified in the space reference frame. Vector $\boldsymbol{\omega}$ is constrained to point on this invariable plane and at the same time on the surface of its constraint ellipsoid. The latter is to be translated in the space reference frame, in which, generally, it is constantly rotating. Combining these two constraints bears the intersection of the invariable plane γ and the constraint ellipsoid $\varepsilon_{\boldsymbol{\omega}}$. Where do they intersect? It is suitable to note that the gradient of E_{Krot}, in respect to $\boldsymbol{\omega}$, is in the direction of $\mathbf{L}$:

$$\nabla_{\boldsymbol{\omega}} E_{Krot} = \frac{dE_{Krot}}{d\boldsymbol{\omega}} = \frac{d\left(\mathbf{L}\boldsymbol{\omega}^{\sim}\right)}{2d\boldsymbol{\omega}} = \frac{1}{2}\left(\frac{d\mathbf{L}}{d\boldsymbol{\omega}}\boldsymbol{\omega}^{\sim} + \mathbf{L}\frac{d\boldsymbol{\omega}^{\sim}}{d\boldsymbol{\omega}}\right) = \frac{1}{2}\mathbf{L}$$

The gradient is in the direction of the normal vector $\mathbf{n}_{\varepsilon_{\boldsymbol{\omega}}}$ to the ellipsoid $\varepsilon_{\boldsymbol{\omega}}$ surface, because the ellipsoid surface is the invariant of E_{Krot}. The projection of $\nabla_{\boldsymbol{\omega}} E_{Krot}$ over the ellipsoid surface is zero. At the same time, the normal vector of the invariable plane is in the direction of $\mathbf{L}$ as well. It becomes obvious that at the point of intersection, both the invariable plane and the constraint ellipsoid have coinciding normal vectors. Hence, the invariable plane and the ellipsoid intersect tangentially, i.e. the ellipsoid always touches the plane and rolls over it. The rolling occurs without slipping, because the point of tangential intersection lies on the axis of rotation (vector $\boldsymbol{\omega}$) and hence has zero rotational velocity – does not slip over the invariable plane, which is stationary. This constraint construction is the well-known Poinsot construction (Fig. 2). The curve drawn by vector $\boldsymbol{\omega}$ on γ is called herpolhode and on $\varepsilon_{\boldsymbol{\omega}}$ - polhode. The herpolhode and the polhode always touch tangentially where vector $\boldsymbol{\omega}$ points.

This construction yields good understanding and visualization of vector $\boldsymbol{\omega}$ and presents the bases for constraint manipulation. But it, solely, does not solve the stabilization problem in the described simulation, because the stored vector in the body state array is $\mathbf{L}$, not $\boldsymbol{\omega}$. It is more accurate and would lead to less error accumulation if the constraint construction, used for error correction, is over vector $\mathbf{L}$, instead of over vector $\boldsymbol{\omega}$. A similar construction over $\mathbf{L}$ is needed.

Such a construction is defined by transferring the constraint over $\mathbf{L}$ in the body reference frame. $\mathbf{L}$ is constant, but once transferred into the body frame, as $\mathbf{L}'$, it is no longer constant and, generally, continuously rotates such that $\mathbf{L}' = \mathbf{L}\mathbf{R}^{\sim}$. But it is still constrained in its magnitude, because rotation does not change magnitude. Vector $\mathbf{L}'$ has constant length and is restricted to point on the surface of a sphere σ' with radius L (Fig. 3). It follows that vector $\mathbf{L}'$ should point on the intersection of this constraint sphere σ' and its constraint ellipsoid $\varepsilon_{\mathbf{L}}'$. This intersection is shown on Fig. 3 in the first octant.

Ellipsoid $\varepsilon_{\mathbf{L}}'$ constraint:

$$\mathbf{L}'\mathbf{I}'^{-1}\mathbf{L}'^{\sim} = L_x'^2 I_{xx}'^{-1} + L_y'^2 I_{yy}'^{-1} + L_z'^2 I_{zz}'^{-1} = 2E_{Krot} = const \quad (5)$$

Sphere σ' constraint:

$$|\mathbf{L}| = |\mathbf{L}'| = L \Rightarrow L_x'^2 + L_y'^2 + L_z'^2 = L^2 = const \quad (6)$$

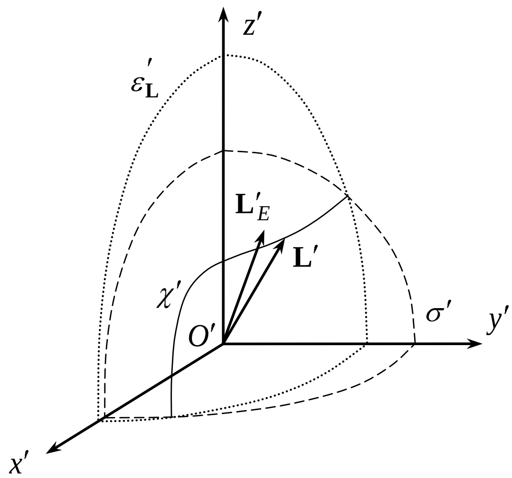

Figure 3. Constraint construction over $\mathbf{L}'$ in the body reference frame (first octant).

The above two equations constraint vector $\mathbf{L}'$ to point on the curve $\chi' = \varepsilon_{\mathbf{L}}' \wedge \sigma'$, representing the intersection of $\varepsilon_{\mathbf{L}}'$ and σ' in the body reference frame. It is immediately visible that this movement has one degree of freedom. If the current value of vector $\mathbf{L}'$ has accumulated error, this vector would, in general, deviate from curve χ'. Let us denote the erroneous vector with $\mathbf{L}'_E$ (Fig. 3). To correct this vector and make it point back on curve χ' again, a correction to the orientation of the rigid body is needed. If the most appropriate correction, back to the curve, is found, then the correcting rotation is easily extracted using vectors $\mathbf{L}'$ and $\mathbf{L}'_E$. This correcting rotation should transform $\mathbf{L}'_E$ into $\mathbf{L}'$. A criterion for the correction, that seems most suitable, is to minimize the correction angle, i.e. minimize the angle between $\mathbf{L}'_E$ and $\mathbf{L}'$. This criterion leads to the following equation:

$$\mathbf{L}_E\mathbf{L}'^{\sim} = L^2 \cos\beta = L'_{Ex}L'_x + L'_{Ey}L'_y + L'_{Ez}L'_z \to \max \quad (7)$$

while $\mathbf{L}'$ should satisfy equations (5) and (6). The system of equations (5), (6) and (7), once solved, leads to an analytical solution to the stabilization problem.

Another approach is to solve the system approximately by generating a correction parameter on each integration step of the simulator. On Fig. 3 vector $\mathbf{L}'_E$ points inside the ellipsoid, so in this example the erroneous energy level will be less than the proper energy constant E_{Krot}. On each integration step, vector $\mathbf{L}'_E$ should be "pushed" with a small correcting amount against a higher energy level, thus approaching curve χ' in the fastest possible way by the smallest displacement. This direction coincides with the gradient of the kinetic energy of rotational motion $\nabla_{\mathbf{L}'}E_{Krot}$ in respect to vector $\mathbf{L}'$. Changing vector $\mathbf{L}'$ in this direction yields the fastest growth of E_{Krot}. From equation (4) and taking into account equation (1) follows:

$$\nabla_{\mathbf{L}'}E_{Krot} = \frac{dE_{Krot}}{d\mathbf{L}'} = \frac{d\left(\mathbf{L}'\mathbf{I}'^{-1}\mathbf{L}'^{\sim}\right)}{2d\mathbf{L}'} = \frac{d\left(L'^2_x I'^{-1}_{xx} + L'^2_y I'^{-1}_{yy} + L'^2_z I'^{-1}_{zz}\right)}{2d\mathbf{L}'} = \left(L'_x I'^{-1}_{xx}, L'_y I'^{-1}_{yy}, L'_z I'^{-1}_{zz}\right) = \boldsymbol{\omega}'$$

But vector $\mathbf{L}'$ is constrained on the sphere σ', so its correction change should be in the direction of $\nabla_{\mathbf{L}'}E_{Krot}$ projection over σ as follows:

$$\nabla_{S\mathbf{L}'}E_{Krot} = \nabla_{\mathbf{L}'}E_{Krot} - \mathbf{n}_\sigma(\nabla_{\mathbf{L}'}E_{Krot}\mathbf{n}_\sigma^{\sim}) = \boldsymbol{\omega}' - \frac{\mathbf{L}'(\boldsymbol{\omega}'\mathbf{L}'^{\sim})}{L^2} = \mathbf{L}'\mathbf{I}'^{-1} - \mathbf{L}'\frac{2E_{Krot}}{L^2}$$

taking into account that the normal vector of sphere σ' is parallel to its radius, which in fact is $\mathbf{L}'$. Thus $\mathbf{n}_\sigma = \frac{\mathbf{L}'}{L}$. The following equation is also considered:

$$2E_{Krot} = \boldsymbol{\omega}\mathbf{L}^{\sim} = \boldsymbol{\omega}'\mathbf{R}\left(\mathbf{L}'\mathbf{R}\right)^{\sim} = \boldsymbol{\omega}'\mathbf{R}\mathbf{R}^{\sim}\mathbf{L}'^{\sim} = \boldsymbol{\omega}'\mathbf{L}'^{\sim}$$

The stabilization algorithm then utilizes the surface gradient $\nabla_{S\mathbf{L}'}E_{Krot}$ to adjust $\mathbf{L}'_E$ on each integration step by applying a rotation correction to the angular velocity vector.

The correction parameter is varied exponentially, i.e. implementing an exponential feedback in order to cope successfully with states generating fast arousing errors and, on the other hand, to guarantee smooth correction in the most common states of slow arousing errors. A block diagram of the integration step is presented on Fig. 4.

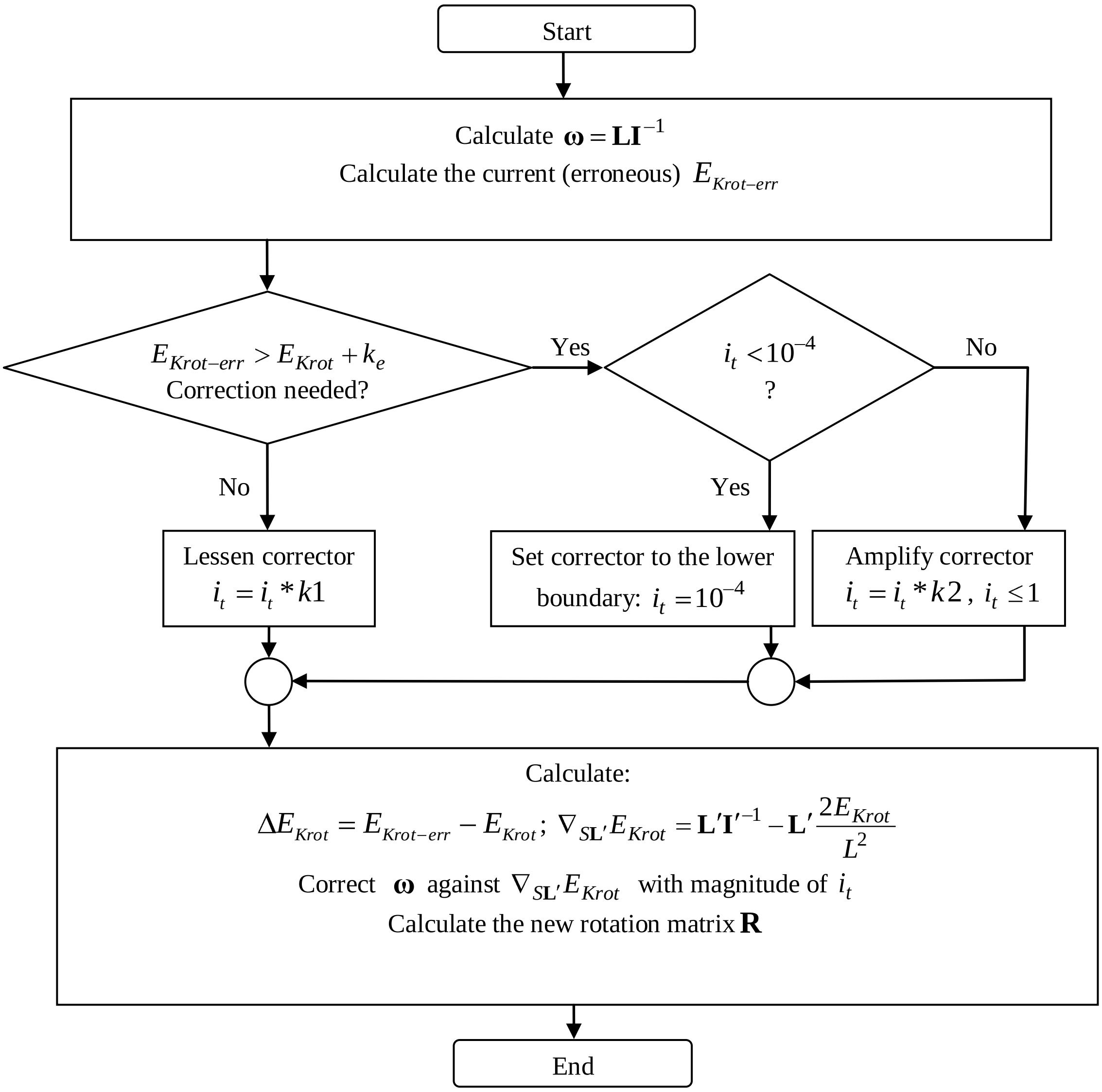

Figure 4. Stabilization algorithm block diagram – one integration cycle.

IV. CONCLUSION

The simulation of rigid body motion gives the opportunity to test scenarios of unmanned aerial vehicles and satellite motion and thus study the behavior of these devices in free space or in the air. The researcher, using the simulation, is presented with insights that are impossible to attain in laboratory conditions. The stabilization of the simulated processes guarantees that the experiment process will obey the investigated physical laws.

V. REFERENCES

[1] R. Outbib, and J. Vivalda, "Stabilization of the rigid body about the middle axis," Proceedings of the 1st Mediterranean Symposium, June 21 - June 23 1993

[2] P. Gurfil, "Stabilization of Rigid Body Dynamics using the Serret-Andoyer Variables," American Control Conference, June 8-10, 2005. Portland, OR, USA

[3] T. Fukushima, "Efficient Integration of Torque-Free Rotation by Energy Scaling Method," Proceedings of the "Journées Systèmes de Référence Spatio-Temporels 2005", Warsaw, 19-21 September 2005

[4] H. Goldstein, C. Pool, and J. Safko, Classical mechanics, third edition. Addison Wesley, Miami

[5] U. Ascher, H. Qin, and S. Reich, "Stabilization of DAEs and invariant manifolds," Technical Report: TR-92-17, University of British Columbia Vancouver, BC, Canada, Canada, 1992

[6] J. Baumgarte, "Stabilization of constraints and integrals of motion in dynamical systems," Computer Methods in Applied Mechanics and Engineering, (1):1–16, 1972.

[7] V. Arnold, Mathematical Methods of Classical Mechanics, Second Edition, Springer-Verlag, 1989